GENERAL APPROXIMATION THEORY

IN NORMED LINEAR SPACES

A THESIS

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ABSTRACT

MATHEMATICS

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General Approximation Theory in Normed Linear Spaces

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In this thesis the concept of the general approximation theory is investigated. In the first chapter, the baisc definitions and properties of normed linear spaces and related theorems used in subsequent chapters are discussed. In the second and third chapters as a main thrust of this study, the author discusses various approximation theory results.

In the final chapter, two important applications of the approximation theory are presented.

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CHAPTER I

BASIC CONCEPTS

In this chapter, we discuss the elementary results of Function Analysis. These results are used in development of the Approximation Theory in subsequent chapters.

Most of the results are proved and terms are defined. However, because of the size of this thesis the proofs of some of the theorems are omitted and references are provided.

1.1 Properties of Normed Linear Spaces

Definition 1.1.1

A vector space X over a scalar field F, is a set of elements called vectors such that for $x,y \in X$ and $a \in F$ there are unique vectors $x+y$ and $a \cdot x \in X$ with the following properties:

i) $a(x+y) = ax+ay$, $(a+b)x = ax+bx$ for $a,b \in F$ and $x,y \in X$

ii) X is an abelian group under addition.

iii) $(ab)x = a(bx)$, $1.x = x$ where 1 is the identity element of F under product.

Unless stated otherwise $F = R^2$ in what follows.

Definition 1.1.2

A metric space is a pair (X,d) , where X is a set and d is a function defined on $X \times X$ such that for all $x,y,z \in X$ the following properties hold:

i) d is a real finite and nonnegative.

ii) $d(x,y) \geqslant o$ and $d(x,y) = o$ if and only if $x = y$

iii) $d(x,y) = d(y,x)$

iv) $d(x,y) \leqslant d(x,z) + d(z,y)$

<u>Definition 1.1.3</u>

A normed linear space X is a linear space with the norm function $\|.\|$ defined on it such that for all $x,y \in X$ the following properties hold:

i) $\|x\| \geqslant o$ and $\|x\| = o$ if and only if $x = o$

ii) $\|ax\| = |a|\,\|x\|$ for all $a \in F$ and $x \in X$

iii) $\|x+y\| \leqslant \|x\| + \|y\|$ for $x,y \in X$

<u>Definition 1.1.4</u>

A normed linear space is said to be complete or Banach if every Cauchy sequence in X converges to a point in X.

<u>Definition 1.1.5</u>

An inner product space X is a linear space together with a mapping $\langle .,. \rangle$ from XxX into the scaler field F such that for all $x,y,z \in X$ and $a \in F$ the following properties hold:

i) $\langle x,x \rangle \geqslant o$ and $\langle x,x \rangle = o$ if and only if $x=o$

ii) $\langle x,y \rangle = \overline{\langle y,x \rangle}$

iii) $\langle ax,y \rangle = a\langle x,y \rangle$

iv) $\langle x+y,z \rangle = \langle x,z \rangle + \langle y,z \rangle$

<u>Remark 1.1.1</u>

a) An inner product on space X defines a norm on X by

$$\|x\| = \sqrt{\langle x,x \rangle}$$

b) A complete inner product space is called a Hilbert space.

c) Every Hilbert space is Banach space but not vice versa.

Definition 1.1.6

Let X , Y be two normed spaces.

a) A mapping U is said to be isometric if for all $x \in X$ we have

$$\| x \|_X = \| U(x) \|_Y$$

b) The space X is said to be isometric with Y if there exists a bijective isometric mapping from X onto Y.

Theorem 1.1.1

For any normed space X there exists a complete normed space W, called the completion of X with a subspace Y that is isometric with X, and is dense in W. The space W is unique except for isometries.

Proof:

Reader is advised to refer to Introductory Functional Analysis by Kreyszig, Wiley 1977.

Definition 1.1.7

a) A nonempty set M in a metric space (X,d) is bounded if the diameter

$$\delta(M) = \sup_{x,y \in M} d(x,y) < \infty .$$

b) A nonempty set M in a normed space $(X , \| . \|)$ is called bounded if and only if there exists a positive number $c < \infty$ such that $\| x \| \leq c$ for all $x \in M$.

c) A set $E \subset X$ is called compact if we can select from any sequence $\{ x_n \}$ of points of E a subsequence that converges to a point in E.

Theorem 1.1.2

In a complete normed space X, if the series

$\sum_{i=1}^{\infty} \|x_i\|$ converges then the series $\sum_{i=1}^{\infty} x_i$ converges.

Proof:

Let $Sn = \sum_{i=1}^{n} \|x_i\|$, $Sm = \sum_{i=1}^{m} \|x_i\|$

and

$Zn = \sum_{i=1}^{n} x_i$, $Zm = \sum_{i=1}^{m} x_i$.

Since $\sum_{i=1}^{\infty} \| x_i \|$ converges, the sequence $\{ Sn \}$ converges. Hence, for each $\varepsilon > 0$ there exists a positive integer N such that $| Sn-Sm | < \varepsilon$ whenever $m,n > N$.

Now
$$\| Zn-Zm \| = \left\| \sum_{i=m+1}^{n} x_i \right\|$$

$$\leqslant \left| \sum_{i=m+1}^{n} \|x_i\| \right|$$

$$= \| Sn-Sm \| < \varepsilon \quad \text{whenever} \quad m,n > N.$$

Since X is complete, the sequence $\{ Zn \}$ converges to an element $Z \in X$. In other words,

$$\sum_{i=1}^{\infty} x_i = \lim_{n \to \infty} Zn = Z.$$

1.2 Examples of normed spaces

Example 1.2.1

Let C $[a,b]$ be the space of all real-valued functions defined and continuous on a given closed interval $J= [a,b]$.

i) The space is metric space with metric d given by

$$d(x,y) = \max_{t \in J} |x(t) - y(t)| \quad \text{for all } x,y \in C\,[a,b].$$

ii) $C\,[a,b]$ is normed space with norm defined by

$$\|x\| = \max_{t \in J} |x(t)| \quad \text{for all } x \in C\,[a,b].$$

iii) $C\,[a,b]$ is a Banach space.

iv) $C\,[a,b]$ is not an inner product space.

Example 1.2.2

Let $L^2\,[a,b]$ be the space of all real-valued functions defined on the interval $J = [a,b]$ such that if

$$x \in L^2\,[a,b] \text{ then } \left[\int_a^b |x(t)|^2 \right]^{\frac{1}{2}} < \infty .$$

i) The space $L^2\,[a,b]$ is a metric space with

$$d(x,y) = \left[|x(t)-y(t)|^2 \right]^{\frac{1}{2}}.$$

ii) The space $L^2\,[a,b]$ is a normed space with norm

$$\|x\| = \left[\int_a^b |x(t)|^2 \right]^{\frac{1}{2}}.$$

iii) The space $L^2\,[a,b]$ is a Banach space.

iv) The space $L^2\,[a,b]$ is a complete inner product space with inner product given by

$$\langle x,y \rangle = \int_a^b x(t)\,\overline{y(t)}\,dt .$$

Hence, it is a Hilbert space.

Example 1.2.3

Let l^{∞} be the space of all sequences

$$x=(u_1,u_2,u_3, \ldots .)$$

of complex numbers such that

$$|u_j| \leqslant Cx \quad \text{for all } j\text{'s.}$$

i) The space l^{∞} is a metric space with

$$d(x,y) = \sup_{j \geqslant 1} |u_j - v_j|$$

where $x=(u_1,u_2,u_3, \ldots .)$ and $y=(v_1,v_2,v_3, \ldots)$.

ii) The space l^{∞} is a complete normed space with the norm

$$\|x\| = \sup_{j \in N} |u_j|.$$

iii) The space l^{∞} is not an inner product space.

Example 1.2.4

Let l^{p}-space be the space of all sequences $x=(u_1,u_2,u_3,\ldots)$ of numbers such that

$$\sum_{j=1}^{\infty} |u_j|^p < \infty .$$

i) The space l^p is a metric with

$$d(x,y) = \left[\Sigma |u_j - v_j|^p\right]^{\frac{1}{p}}.$$

ii) The space is a complete normed space with norm

$$\|x\| = \left[\sum_{j=1}^{\infty} |u_j|^p\right]^{\frac{1}{p}}.$$

iii) The space l^p with $P \neq 2$ is not an inner product space. If p=2 then it is Hilbert space with inner product given by

$$\langle x,y \rangle = \sum_{j=1}^{\infty} u_j \bar{v}_j \ .$$

In case the sequences are finite the spaces are denoted by l_n^{∞} and l_n^{p} .

<u>Theorem 1.2.1</u>

Let $\{u_j\}_1^n$ and $\{v_j\}_1^n$ be two sequences of any numbers. Let p and q be any two real positive numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Then the following inequality holds.

$$\sum_{j=1}^{n} |u_j v_j| \leqslant \left[\sum_{j=1}^{n} |u_j|^p\right]^{\frac{1}{p}} \cdot \left[\sum_{j=1}^{n} |v_j|^q\right]^{\frac{1}{q}} .$$

This is called Hölder inequality.

<u>Proof</u>

The proof is omitted here, and the reader is advised to refer to Functional Analysis by Kantorovich and Akilov, Pergamon Press, 1964.

<u>Remark 1.2.1</u>

If the series $\sum_{j=1}^{\infty} |u_j|^p$ and $\sum_{j=1}^{\infty} |v_j|^q$ converge,

then $$\sum_{j=1}^{\infty} |u_j v_j| \leqslant \left[\sum_{j=1}^{\infty} |u_j|^p\right]^{\frac{1}{p}} \cdot \left[\sum_{j=1}^{\infty} |v_j|^q\right]^{\frac{1}{q}} .$$

1.3 Linear Operators

Definition 1.3.1

Let X,Y be linerar spaces. The $U:X \longrightarrow Y$ is called a linear operator if for all x,y in X and $a \epsilon R^2$, we have

$$U(x+y) = U(x) + U(y) \quad \text{and} \quad U(ax) = aU(x).$$

Definition 1.3.2

Let X and Y be two normed spaces and let $U:X \longrightarrow Y$ be a linear operator. Then U is said to be bounded if there exists k such that

$$\|U(x)\| \leqslant k \|x\| \quad \text{for all } x \epsilon X.$$

Definition 1.3.3

Let $U:X \longrightarrow Y$ be an operator (linear or not) and let X,Y be two normed spaces. Then U is said to be continuous at $x_o \epsilon X$ if for each $\varepsilon > o$ there is $\delta > o$ such that

$$\|U(x)-U(x_o)\| < \varepsilon \quad \text{whenever } \|x-x_o\| < \delta .$$

U is said to be continuous if it is continuous at every point $x \epsilon X$.

Definition 1.3.4

Let $U:X \longrightarrow Y$ be a linear operator, where X and Y are normed spaces. Then U is said to be relatively compact operator if for every bounded subset M of X, $\overline{U(M)}$ is compact. It is compact or completely continuous if U(M) is compact.

Definition 1.3.5

Let $U:X \longrightarrow Y$ be a linear operator, where X and Y are normed spaces. Then the norm on U is defined by

$$\| U \| = \sup_{\substack{x \in X \\ \|x\| \neq 0}} \frac{\| U(x) \|}{\| x \|} .$$

Remark 1.3.1

If U is bounded then

$$\| U(x) \| \leqslant \| U \| \; \| x \|$$

with $\| U \| = \min \{ k : \; \| U(x) \| \leqslant k \; \| x \| \text{ for all } x \in X \}$.

Definition 1.3.6

A linear functional f on X is a linear operator $f: X \longrightarrow R^2$.

Theorem 1.3.1

Let $U: X \longrightarrow Y$ be a linear operator, where X and Y are normed spaces. Then

i) U is continuous if and only if U is bounded.

ii) If U is continuous at a point $x_o \in X$ then U is continuous everywhere.

Proof

i) Let $x_o \in X$. If U is bounded then there exists $k > o$ such that

$$\| U(x) - U(x_o) \| = \| U(x - x_o) \| \leqslant k \; \| x - x_o \| .$$

For $\varepsilon > o$, let $\delta = \frac{\varepsilon}{k}$. If $\| x - x_o \| < \delta$ then $\| U(x) - U(x_o) \| < \varepsilon$. This shows the continuity of U. Conversely, let U be continuous. If U is not bounded then there exists a sequence x_n such that

$$\| U(x_n) \| > n \; \| x_n \| \quad \text{for } n \geqslant 1 .$$

Note $x_n \neq o$ for any n. If $z_n = \dfrac{x_n}{n\|x_n\|}$ then from above relation we have

$$\| U(z_n)\| = \frac{\|U(x_n)\|}{n\,\|x_n\|} > 1.$$

Moreover, $\| Z_n \| = \frac{1}{n} \longrightarrow o$. This shows that U is not continuous at o.

ii) We simply show that if U is continuous at x_o, it is bounded. For $\varepsilon > o$, there exists $\delta > o$ such that

$$\| U(x)-U(x_o)\| < \varepsilon \quad \text{if} \quad \| x-x_o\| < \delta .$$

Let $y \in X$ such that $y \neq 0$. Let $x = \dfrac{\delta\, y}{2\|y\|} + x_o$ so that

$\| x - x_o \| = \dfrac{\delta}{2} < \delta$. Consequently,

$$\| U(x) - U(x_o)\| < \varepsilon. \text{ Hence } \frac{\delta}{2\,\|y\|}\,\|U(y)\| < \varepsilon .$$

This implies $\| U(y)\| < k\,\|y\|$ for all y, where $k = \dfrac{2\,\varepsilon}{\delta}$.

Theorem 1.3.2

$$\| U \| = \sup_{\substack{x \in X \\ \|x\| \neq o}} \frac{\|U(x)\|}{\|x\|} = \sup_{\substack{x \in X \\ \|x\| = 1}} \| U(x) \|$$

Proof

It is trivial and hence omitted.

Theorem 1.3.3

Let $U : X \longrightarrow Y$ be linear operator, where X and Y are normed spaces. Then U is continuous at $x_o \in X$ if and only if for each sequence $\{ x_j \}$ in X that converges to x_o, the sequence $\{U(x_j)\}$ converges to $U(x_o)$.

Proof:

If U is bounded and $x_n \longrightarrow x_o$ then there exists $k > o$ such that

$$\| U(x_n) - U(x_o)\| \leqslant k \|x_n - x_o\| \text{ for } n \geqslant 1.$$

This shows that $U(x_n) \longrightarrow U(x_o)$.

Conversely, let the condition be satisfied. If U is not bounded then there exists x_n such that

$$\|U(x_n)\| > n \|x_n\| \text{ for } n \geqslant 1. \quad \text{Note } x_n \neq o \text{ for } n \geqslant 1.$$

Now if $z_n = \dfrac{x_n}{n\|x_n\|}$ then $\|z_n\| = \dfrac{1}{n} \longrightarrow o$, and $\|U(z_n)\| > 1$.

Thus, $U(z_n) \not\longrightarrow o$. This contradicts the hypothesis.

Theorem 1.3.4

Let $U_2 : X \longrightarrow Y$, $U_1 : Y \longrightarrow Z$ and $U : X \longrightarrow X$ be bounded linear operators, where X, Y and Z are normed spaces. Then

$$\|U_1 U_2\| \leqslant \|U_1\| \, \|U_2\| \text{ and}$$

$$\|U^n\| \leqslant \|U\|^n, \; n \in N.$$

Proof:

It is easy to verify that $U_1 U_2 : X \to Z$ is linear bounded operator if U_2, U_1 are bounded and linear. Moreover,

$$\|(U_1 U_2)\, x\| = \|U_1(U_2(x))\|$$

$$\leqslant \|U_1\| \, \|U_2(x)\| \leqslant \|U_1\| \, \|U_2\| \|x\| .$$

Thus,

$$U_1 U_2 = \sup_{\|x\|=1} \|(U_1 U_2)\, x\|$$

$$\leqslant \sup_{\|x\|=1} \|U_1\| \, \|U_2\| \, \|x\| = \|U_1\| \, \|U_2\|.$$

To prove $\|U^n\| \leqslant \|U\|^n$, we use the first part with $U_1 = U_2$ and then induction.

Definition 1.3.7

Let X be a normed space. Then the normed linear space X' of all bounded linear functionals on X with the norm for $f \epsilon X'$ defined by

$$\|f\| = \sup_{\substack{x \epsilon X \\ \|x\| \neq o}} \frac{|f(x)|}{\|x\|} = \sup_{\substack{x \epsilon X \\ \|x\| = 1}} |f(x)|$$

is called the dual space of X.

Definition 1.3.8

Let $\{fn\}$ be a sequence of bounded linear functionals on a normed space X. Then

i) $\{fn\}$ is strongly convergent if there is $f \epsilon X'$ such that $\|fn - f\| \longrightarrow o$.

ii) $\{fn\}$ is weakly convergent if there is $f \epsilon X'$ such that $|f_n(x) - f(x)| \longrightarrow o$ for all $x \epsilon X$.

In this case we write $f_n \xrightarrow{w} f$.

Next two results are used in the development of the approximation theory later. The results are mentioned without proofs for which references may be made to Functional Analysis by Kantorovich and Akilov, Pergamon Press, 1964.

Theorem 1.3.5

Let $U: X \longrightarrow Y$ be a linear operator, where X and Y are normed spaces. If the equation $U(x) = y$ has a solution for each $y \in Y$ and there exists a positive number m such that

$\|U(x)\| \geq m \; \|x\|$ for every $x \in X$, then U has linear inverse U^{-1} which satisfies

$$\|U^{-1}\| \leq \frac{1}{m} .$$

Theorem 1.3.6

A necessary condition for a subset E of a normed linear space X to be compact is that for every sequence $\{fn\}$ of bounded linear functionals converging weakly to zero, we have $f_n(x) \xrightarrow[n \to \infty]{} o$ uniformly on E.

1.4 Linear Operators in a Hilber Space

Definition 1.4.1

i) A vector space X is said to be direct sum of two subspace Y and Z if each $x \in X$ has a unique representation

$x = y + z$ with $y \in Y$, $z \in Z$.

We represent this fact by $X = Y \oplus Z$ and call Z an algebric complement of Y in X.

ii) The orthognal complement of Y is defined by

$Y^{\perp} = \{ z \in X / z \perp Y \}$

Definition 1.4.2

a) Let X be a normed linear space and $P: X \longrightarrow X$ be a linear operator such that $P^2 = P$. Then P is called a projection.

b) If X is a Hilbert space and the range and the null spaces of the projection P are orthognal, then P is called an orthognal projection.

Remark 1.4.1

If p is projection on a normed linear space X, then the following bold.

a) The range space of P is the set of fixed points.

b) The space X is the direct sum of the range and the null spaces.

Definition 1.4.3

a) Let $U:H_1 \longrightarrow H_2$ be a bounded linear operator, where H_1, H_2 are Hilbert spaces. Then the operator $U^*:H_2 \longrightarrow H_1$ such that $\langle U(x), y \rangle = \langle x, U^*(y) \rangle$ is called Hilbert-adjoint of U.

b) Let $U:H \longrightarrow H$ be linear operator where H is a Hilbert space. Then U is said to be

i) Self-adjoint if $U^* = U$,

ii) Unitary if U is bijective and $U^* = U^{-1}$,

iii) Normal if $UU^* = U^*U$.

Remark 1.4.2

If U is self-adjoint then $\langle U(x), y \rangle = \langle x, U(y) \rangle$.

Theorem 1.4.1

Let $P:H \longrightarrow H$ be a projection, where H is a Hilbert space. Then P is orthogonal if and only if P is self-adjoint.

Proof:

Let P be an orthogonal projection so that its range space R(P) and N(P) are orthogonal and $X = R(P) \oplus N(P)$. Let $x, y \in X$, then from the projection theorem there exist unique representations

$x = x_1 + x_2$ and $y = y_1 + y_2$ where $x_1, y_1 \in R(P)$ and $x_2, y_2 \in N(P)$.

Noting that $P(x_1) = x_1$, $P(y_1) = y_1$, we have

$$\langle P(x),y\rangle = \langle P(x_1),y_1\rangle + \langle P(x_1),y_2\rangle = \langle x_1,y_1\rangle$$

and

$$\langle x,P(y)\rangle = \langle x_1,P(y_1)\rangle + \langle x_2,P(y_1)\rangle = \langle x_1,y_1\rangle .$$

Hence $\langle P(x),y\rangle = \langle x,P(y)\rangle$. This shows that P is self-adjoint.

Conversely, let P be self-adjoint. If $x \in R(P)$ and $y \in N(P)$ then

$$\langle x,y\rangle = \langle P(x),y\rangle = \langle x,P(y)\rangle = \langle x,o\rangle = 0 .$$

This shows that $R(P) \perp N(P)$ and hence P is orthogonal.

Definition 1.4.4

Let $U: l^p \longrightarrow l^r$ be a linear operator where U is determined by the matrix :

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} & \cdots \\ a_{12} & a_{22} & \cdots & a_{2k} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ik} & \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} .$$

Let $x = (\xi_1, \xi_2, \xi_3, \ldots) \in l^p$, $y = (\eta_1, \eta_2, \eta_3, \ldots) \in l^r$.

We put the following notations:

$$[x]_n = (\xi_1, \xi_2, \xi_3, \ldots, \xi_n, o, \ldots), [y]_n = (\eta_1, \eta_2, \eta_3, \ldots, \eta_n, o, \ldots).$$

$$[a_{ik}]_n = \begin{cases} a_{ik} & \text{if } i \leqslant n \\ o & \text{if } i > n \end{cases}, \quad [a_{ik}]_{mn} = \begin{cases} a_{ik} & \text{if } i \leqslant n, \ k \leqslant m \\ o & \text{if } i > n \text{ or } k > m \end{cases}$$

and

$$[A]_n = \begin{bmatrix} [a_{11}]_n & [a_{12}]_n & \cdots & [a_{1k}]_n & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ [a_{i1}]_n & [a_{i2}]_n & \cdots & [a_{ik}]_n & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{ik} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} & \cdots \\ 0 \ldots 0 & & \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$[A]_{nm} = \begin{bmatrix} [a_{12}]_{nm} & [a_{12}]_{nm} & \cdots & [a_{1k}]_{nm} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ [a_{i1}]_{nm} & [a_{i2}]_{nm} & \cdots & [a_{ik}]_{nm} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Let U_n and U_{nm} be linear operators determined by $[A]_n$ and $[A]_{nm}$ respectively.

Theorem 1.4.2

A necessary and sufficient condition for the linear operator U to be completely continuous is that

$$U_n \xrightarrow[n \to \infty]{} U \quad \text{or} \quad U_{nm} \xrightarrow[n,m \to \infty]{} U .$$

In other words, either

$$\lim_{n \to \infty} \|U_n - U\| = 0 \quad \text{or} \quad \lim_{n,m \to \infty} \|U_{nm} - U\| = 0.$$

Proof

A reference may be made to Functional Analysis by Kantorovich and Akilov, Pergamon Press, 1964.

<u>Definition 1.4.5</u>

Let $U: X \longrightarrow X$ be a linear operator from a normed space X to itself. Then the equation of the form

$$x - \lambda U(x) = y \ , \ x, y \in X$$

is called equation of the second kind.

<u>Definition 1.4.6</u>

A bounded periodic function $x(t)$, of period 1, is a function associated with an integral over the interval $[o,1]$ i.e., $\int_o^1 x(t)\,dt$ such that the following conditions are satisfied:

i) $\int_o^1 [a x_1(t) + b x_2(t)]\,dt = a \int_o^1 x_1(t)\,dt + b \int_o^1 x_2(t)\,dt$,

a, b are constants.

ii) If $x(t) \geqslant o$ in $[o,1]$ then $\int_o^1 x(t)\,dt \geqslant o$.

iii) $\int_o^1 x(t+t_o)\,dt = \int_o^1 x(t)\,dt$, t_o is any real number .

iv) $\int_o^1 x(1-t)\,dt = \int_o^1 x(t)\,dt$.

v) If $x_o(t) \equiv 1$ then $\int_o^1 x_o(t)\,dt = 1$.

CHAPTER II

GENERAL APPROXIMATION THEORY FOR THE EQUATIONS OF SECOND KIND

2.1 The Approximate and Exact Equations

Let $\hat{X}$ be a complete subspace of a normed space X. Let P be an operator projecting X onto $\hat{X}$ such that $P(X)=\hat{X}$ and $P^2=P$.

Thus, for all $x \epsilon X$, $P(x) = P^2(x)$ which implies $\hat{x} = P(\hat{x})$ where $P(x) = \hat{x} \epsilon \hat{X}$. Therefore, P does not change the element of $\hat{X}$. Consider the following two equations of the second kind. The first is in the space X representing the exact equation

$$F(x) \equiv x - \lambda U(x) = y. \tag{2.2.1}$$

The second is in the subspace $\hat{X}$ representing the approximate equation

$$\hat{F}(\hat{x}) \equiv \hat{x} - \lambda \hat{U}(\hat{x}) = P(y). \tag{2.1.2}$$

Here U and $\hat{U}$ are bounded linear operators in X and $\hat{X}$ respectively and $x \epsilon X$, $\hat{x} \epsilon \hat{X}$.

Definition 2.1.1

Let U, $\hat{U}$ and P be defined as above. If for every $\hat{x} \epsilon \hat{X}$ the inequality

$$\| PU(\hat{x}) - \hat{U}(\hat{x}) \| \leqslant \alpha \ \| \hat{x} \| \tag{2.1.3}$$

is satisfied, then U and $\hat{U}$ are said to be neighboring operators, where a is independent of $\hat{x}$.

Theorem 2.1.1

The inequality (2.1.3) is equivalent to

(2.1.4) $$\| PF(\hat{x})-\hat{F}(\hat{x}) \| \leqslant |\lambda| a \cdot \| \hat{x} \|$$

where F and $\hat{F}$ are linerar operators defined by (2.1.1) and (2.1.2) respectively.

Proof:

From (2.1.1) we have

$$F(\hat{x})=\hat{x}- \lambda U(\hat{x})$$

which implies

$$PF(\hat{x})=P(\hat{x})-\lambda PU(\hat{x})=\hat{x}- \lambda PU(\hat{x}).$$

From (2.1.2)

$$\hat{F}(\hat{x})=\hat{x}-\lambda\hat{U}(\hat{x}).$$

Thus,

$$PF(\hat{x})-\hat{F}(\hat{x})= \lambda \left[PU(\hat{x})-\hat{U}(\hat{x})\right].$$

Therefore,

$$\| PF(\hat{x})-\hat{F}(\hat{x}) \| = |\lambda| \, \|PU(\hat{x})-\hat{U}(\hat{x})\| \leqslant |\lambda| a \| \hat{x} \|.$$

Conversely, from above we have

$$\lambda \left[PU(\hat{x})- \hat{U}(\hat{x})\right]=PF(\hat{x})-\hat{F}(\hat{x}),$$

which implies

$$|\lambda| \, \| PU(\hat{x})-\hat{U}(\hat{x}) \| = \| PF(\hat{x})-\hat{F}(\hat{x}) \| \leqslant |\lambda| a \| \hat{x} \|.$$

Therefore,

$$\| PU(\hat{x})-\hat{U}(\hat{x}) \| \leqslant a \| \hat{x} \|.$$

Definition 2.1.2

If for $x \epsilon X$ there is $\hat{x} \epsilon \hat{X}$ such that

(2.1.5) $$\| U(x)-\hat{x} \| \leqslant a_1 \| x \|$$

then we say that the elements of the form $U(x)$ can be approximated closely by elements of $\hat{X}$. Here a_1 is independent of x.

Definition 2.1.3

If $y \in X$ is the free term of (2.1.1) and there exists $\hat{y} \in \hat{X}$ such that

(2.1.6) $$\| y-\hat{y} \| \leqslant a_2 \| y \|,$$

then we say that there is a close approximation for the free term of the exact equation. a_2, in general, depends on y.

Theorem 2.1.2

Let $T:X \longrightarrow Y$ be a linear operator, where X and Y are normed spaces. Assume that for each $y \in Y$ there exists an $x \in X$ such that

$$\| T(x)-y \| \leqslant q \| y \| \text{ and } \| x \| \leqslant M \| y \|$$

where $q<1$ and M is constant. Then the equation

(2.1.7) $T(x)=y$

has a solution $x \in X$ satisfying

(2.1.8) $$\| x \| \leqslant \frac{M}{1-q} \| y \| .$$

In other words, solubility of (2.1.7) follows from its approximate solubility.

Proof:

We use the method of exhaushion.

Let $y=y_1 \in Y$. Then by hypothesis there is x_1 in X such that

$$\| T(x_1) -y_1 \| \leqslant q \| y_1 \| \quad \text{and} \quad \| x_1 \| \leqslant M \| y_1 \| .$$

If $y_2=y_1-T(x_1)$, then

$$\|y_2\| = \|T(x_1)-y_1\| \leqslant q \|y_1\| \text{ and } \|x_1\| \leqslant M \|y_1\|.$$

Moreover, for y_2 there is x_2 such that

$$\|y_2-T(x_2)\| \leqslant q \|y_2\| \text{ and } \|x_2\| \leqslant M \|y_2\|.$$

If $y_3=y_2-T(x_2)$, then

$\|y_3\| = \|y_2-T(x_2)\| \leqslant q\|y_2\| \leqslant q^2 \|y_1\|$ and

$\|x_2\| \leqslant M\|y_2\| \leqslant Mq \|y_1\|$.

For y_3 there is x_3 such that

$$\|y_3\| = \|y_2-T(x_2)\| = \|y -T(x_1+x_2)\| \leqslant q^2 \|y_1\|$$

and $$\|x_3\| \leqslant M\|y_3\| \leqslant Mq^2 \|y_1\|.$$

Continuing this way we get

(2.1.9) $\|y_k\| \leqslant q^{k-1} \|y_1\|$; $\|x_k\| \leqslant Mq^{k-1} \|y_1\|$

and

(2.1.10) $y_{n+1}=y_1 -T(\sum_{i=1}^{n} x_i)$ for $n=1, 2, \dots$.

Now

$$\sum_{i=1}^{\infty} \|x_i\| \leqslant \sum_{i=1}^{\infty} Mq^i \|y_1\|$$

$$= M\|y_1\| \sum_{i=1}^{\infty} q^i$$

$$= M \|y_1\| \frac{1}{1-q} \text{, since } 0 \leqslant q < 1.$$

Thus, by theorem 1.1.2, we conclude that $\sum_{i=1}^{\infty} x_i$ converges.

Therefore if $x= \sum_{i=1}^{\infty} x_i$ then

$$\|x\| \leqslant \frac{M}{1-q} \|y_1\|.$$

From (2.1.9), $\lim_{n\to\infty} y_n = o$. Letting $k\longrightarrow\infty$ in (2.1.10) we get $o=y_1-T(x)$. Since $y_1=y$, this gives $T(x)=y$. In other words, x is a solution of (2.1.7) that satisfies (2.1.8).

2.2 The Approximation Theorems

Theorem 2.2.1

Let the inequalities of definitions 2.1.1 and 2.1.2 be satisfied. Assume that the operator F has a two-sides linear inverse and

(2.2.1) $$q=|\lambda|\left[a(1+|\lambda|a_1)+a_1\|PF\|\right]\|F^{-1}\|<1.$$

Then the equation

(2.2.2) $$\hat{F}(\hat{x})=\hat{y}$$

has a solution $\hat{x}^*$ for any $\hat{y}\in\hat{X}$ with

(2.2.3) $$\|\hat{x}^*\|\leqslant\frac{M}{1-q}\|\hat{y}\|$$

and

(2.2.4) $$M=(1+|\lambda|a_1)\|F^{-1}\|.$$

Proof:

It is sufficient to show that the conditions of the theorem 2.1.2 are satisfied for the equation (2.2.2) with the given values of q and M. Consider the equation $F(x)=\hat{y}$, and let x_o be a solution for this equation. Then $x_o=F^{-1}(\hat{y})$. If $x_o=z+\hat{y}$ then (2.1.1) gives

$$F(x_o)=x_o-\lambda U(x_o)=\hat{y}$$
$$=z+\hat{y}-\lambda U(x_o)=\hat{y}.$$

Thus $z=\lambda U(x_o)$ and hence $U(x_o)=\frac{z}{\lambda}$.

From definition 2.1.2 there exists $\hat{x}_o$ such that

$$\| U(x_o)-\hat{x}_o \| \leqslant a_1 \| x_o \| \text{ with } \hat{x}_o = \frac{\hat{z}}{\lambda} .$$

This gives $\| z-\hat{z} \| \leqslant |\lambda| a_1 \| x_o \|$.

Let $\hat{x}=\hat{z}+\hat{y}$. We show that $\hat{x}$ satisfies theorem 2.1.2.

$$\begin{aligned}
\| \hat{F}(\hat{x})-PF(\hat{x}) \| &= \| \hat{F}(\hat{x})-P(\hat{y}) \| = \|\hat{F}(\hat{x})-PF(x_o) \| \\
&\leqslant \| \hat{F}(\hat{x})-PF(\hat{x})\| + \|PF(\hat{x})-PF(x_o)\| \\
&= \| \hat{F}(\hat{x})-PF(\hat{x})\| + \|PF(\hat{x}-x_o)\| \\
&\leqslant |\lambda| a \| \hat{x} \| + \| PF \| \, \| \hat{x}-x_o \| .
\end{aligned}$$

But

$$\begin{aligned}
\| \hat{x}-x_o \| &= \| z-\hat{z} \| \leqslant |\lambda| a_1 \| x_o \| = |\lambda| a_1 \| F^{-1}(\hat{y}) \| \\
&\leqslant |\lambda| a_1 \| F^{-1} \| \, \| \hat{y} \| ,
\end{aligned}$$

$$\begin{aligned}
\| \hat{x} \| \leqslant \| x_o \| + \| \hat{x}-x_o \| &\leqslant \| F^{-1} \| \, \| \hat{y} \| + |\lambda| a_1 \| F^{-1} \| \, \| \hat{y} \| \\
&= (1+|\lambda| a_1) \| F^{-1} \| \, \| \hat{y} \| \\
&= M \| \hat{y} \| \leqslant \frac{M}{1-q} \| \hat{y} \| \quad \text{if } 0 \leqslant q < 1
\end{aligned}$$

and

$$M=(1+|\lambda| a_1) \, \| F^{-1} \| .$$

Moreover, $\| \hat{F}(\hat{x})-PF(\hat{x}) \| = \| \hat{F}(\hat{x})-P(\hat{y}) \| = \| \hat{F}(\hat{x})-\hat{y} \|$.

Therefore,

$$\begin{aligned}
\| \hat{F}(\hat{x})-\hat{y} \| &\leqslant |\lambda| a (1+|\lambda| a_1) \| F^{-1} \| \, \| \hat{y} \| + \| PF \| \, |\lambda| a_1 \| F^{-1} \| \, \| \hat{y} \| \\
&= |\lambda| \, \| F^{-1} \| \left[a(1+|\lambda| a_1) + a_1 \| PF \| \right] \| \hat{y} \| \\
&= q \, \| \hat{y} \| \quad \text{with } q \text{ given by (2.2.1).}
\end{aligned}$$

Thus we have

$$\| \hat{F}(\hat{x})-\hat{y} \| \leqslant q \| \hat{y} \| \quad \text{and} \quad \| \hat{x} \| \leqslant \frac{M}{1-q} \| \hat{y} \|$$

where q and M are defined by (2.2.1) and (2.2.4) respectively.

<u>Corollary 2.2.1</u>

Let the operator $\hat{U}$ be such that the existence of a solution of (2.2.2) for every $\hat{y}\epsilon\hat{X}$ implies its uniqueness. Then given the conditions of theorem 2.2.1, $\hat{F}$ has linear inverse such that

(2.2.5) $$\|\hat{F}^{-1}\| \leq \frac{M}{1-q} .$$

<u>Proof:</u>

By hypothesis of the corollary there is a one-to-one correspondence between the elements $\hat{y}\epsilon\hat{X}$ and the solutions $\hat{x}^*$ of (2.2.2). Thus the existence of $\hat{F}^{-1}$ is guaranteed.

We show that $\hat{F}^{-1}$ is linear and satisfies (2.2.5). Let $\hat{x}_1^*$, $\hat{x}_2^*$ be two distinct solutions of (2.2.2). Then because of one-to-one correspondence there exist $\hat{y}_1$, $\hat{y}_2$ and $\hat{y}_1 \neq \hat{y}_2$ such that $\hat{F}^{-1}(\hat{y}_1) = \hat{x}^*$ and $\hat{F}^{-1}(\hat{y}_2)=\hat{x}_2^*$.

But $\hat{F}(\hat{x}_1^*+\hat{x}_2^*) = \hat{F}(\hat{x}_1*)+\hat{F}(\hat{x}_2^*)=\hat{y}_1+\hat{y}_2$.

Hence, $\hat{F}^{-1}(\hat{y}_1+\hat{y}_2)=x_1^*+x_2^*=\hat{F}^{-1}(\hat{y}_1)+\hat{F}^{-1}(\hat{y}_2)$.

Similarly, $\hat{F}^{-1}(a\hat{y}_1) = a\hat{F}^{-1}(\hat{y}_1)$. This implies $\hat{F}^{-1}$ is linear.

From theorem 2.1.3, $\|\hat{x}^*\| \leq \frac{M}{1-q}\|\hat{y}\|$.

Consequently,

$$\|\hat{F}^{-1}(\hat{y})\| \leq \frac{M}{1-q}\|\hat{y}\| \quad \text{for each } \hat{y}.$$

Now,

$$\|\hat{F}^{-1}\| = \sup_{\|\hat{y}\|=1}\|\hat{F}^{-1}(\hat{y})\| \leq \sup_{\|\hat{y}\|=1}\frac{M}{1-q}\|\hat{y}\| = \frac{M}{1-q}.$$

The following theorem gives the error between the exact and the approximate solutions of equation (2.1.1).

Theorem 2.2.2

Let the inverse of $\hat{F}$ exist and the inequalities of definitions 2.1.1, 2.1.2 and 2.1.3 be satisfied. If (2.1.1) has a solution x^* then

(2.2.6) $\quad \| x^*-\hat{x}^* \| \leqslant p \, \| x^* \|$

where $\hat{x}^*$ is the solution of the equation (2.1.2) and

(2.2.7) $\quad P=2|\lambda| \alpha \, \| \hat{F}^{-1} \| +(\alpha_1 |\lambda| +\alpha_2 \|F\|)(1+ \|\hat{F}^{-1} PF\|).$

Proof:

We show that we can find an $\hat{x}\epsilon\hat{X}$ approximate to x^*. In other words, we show that there exists $\hat{x}\epsilon\hat{X}$ such that

(2.2.8) $\quad \| x^*-\hat{x} \| \leqslant \varepsilon \, \| x^* \|$

where

(2.2.9) $\quad \varepsilon = \min\,(1, \alpha_1 |\lambda| +\alpha_2 \|F\|).$

Take $\hat{y}$, $\hat{z}\epsilon\hat{X}$ such that the inequalities of definitions 2.1.1 and 2.1.2 are satisfied. In other words, since $y=F(x^*)$,

$$\| U(x^*)-\hat{z} \| \leqslant \alpha_1 \| x^* \|$$

and

$$\| y-\hat{y} \| \leqslant \alpha_2 \| y \| \leqslant \alpha_2 \|F\| \, \| x^* \| .$$

Let $\hat{x}=\lambda\hat{z}+\hat{y}$.

From (2.1.1) we have $x^* = \lambda U(x^*)+y$.

Therefore,

$$\begin{aligned} \| x^*-\hat{x} \| &= \|\lambda U(x^*)+y-(\lambda\hat{z}+\hat{y})\| \\ &\leqslant \| \lambda U(x^*)-\lambda\hat{z} \| + \| y-\hat{y} \| \\ &\leqslant |\lambda| \alpha_1 \|x^*\| + \alpha_2 \|F\| \, \|x^*\| \\ &= (\alpha_1|\lambda| +\alpha_2 \|F\|)\|x^*\| . \end{aligned}$$

Moreover, the inequality (2.2.8) is satisfied for $\varepsilon=1$ if we set $\hat{x}=o$.

Thus $\hat{x}$ satisfies $\|x^*-\hat{x}\| \leq \varepsilon \|x^*\|$ with $\varepsilon = \min(1, a_1|\lambda|+a_2\|F\|)$.

If $\hat{x}_o=\hat{F}^{-1}PF(\hat{x})$ then we have

$$\|x^*-\hat{x}^*\| \leq \|x^*-\hat{x}\| + \|\hat{x}-\hat{x}_o\| + \|\hat{x}_o-\hat{x}^*\| .$$

$\|x^*-\hat{x}\|$ is given above and other two terms of the right side are calculated below.

$$\|\hat{x}-\hat{x}_o\| = \|\hat{F}^{-1}\hat{F}(\hat{x})-\hat{F}^{-1}PF(\hat{x})\| = \|\hat{F}^{-1}\left(\hat{F}(\hat{x})-PF(\hat{x})\right)\|$$
$$\leq \|\hat{F}^{-1}\| \; \|\hat{F}(\hat{x})-PF(\hat{x})\|$$
$$\leq \|\hat{F}^{-1}\| \, |\lambda| \, a \, \|\hat{x}\| \quad \text{from theorem 2.1.1.}$$

Now,

$$\|\hat{x}\| \leq \|x^*\| + \|x^*-\hat{x}\| \leq \|x^*\| + \varepsilon\|x^*\| = (1+\varepsilon)\|x^*\| .$$

Hence,

$$\|\hat{x}-\hat{x}_o\| \leq \|\hat{F}^{-1}\| \, |\lambda| \, a \, (1+\varepsilon)\|x^*\| .$$

From theorem 2.2.1 there exists a solution $\hat{x}^*$ for equation (2.2.2) such that $\hat{F}(\hat{x}^*)=\hat{y}$. Then by (2.1.1)

$\hat{F}(\hat{x}^*)=\hat{y}=P(y)=PF(x^*)$ which implies $\hat{x}^*=\hat{F}^{-1}PF(x^*)$.

Thus

$$\|\hat{x}_o-\hat{x}^*\| = \|\hat{F}^{-1}PF(\hat{x})-\hat{F}^{-1}PF(x^*)\|$$
$$= \|\hat{F}^{-1}PF(\hat{x}-x^*)\|$$
$$\leq \|\hat{F}^{-1}PF\| \; \|\hat{x}-x^*\| \leq \|\hat{F}^{-1}PF\| \, \varepsilon \, \|x^*\| .$$

Hence,

$$\|x^*-\hat{x}^*\| \leq \varepsilon\|x^*\| + \|\hat{F}^{-1}\| \, |\lambda| \, a \, (1+\varepsilon)\|x^*\| + \|\hat{F}^{-1}PF\|\varepsilon\|x^*\| .$$

This gives

(2.2.10) $\|x^*-\hat{x}^*\| \leq \left[\|\hat{F}^{-1}\| \, |\lambda| \, a(1+\varepsilon)+\varepsilon(1+\|\hat{F}^{-1}PF\|)\right]\|x^*\|$.

Since $\varepsilon \leq 1$ and $\varepsilon \leq a_1|\lambda|+a_2\|F\|$ (2.2.10) gives

$$\|x^*-\hat{x}^*\| \leq p\|x^*\| \quad \text{where } p \text{ is given by (2.2.7).}$$

Remark 2.2.1

If $p<1$ then $\|x^*-\hat{x}^*\| \leqslant \frac{1}{1-p}\|\hat{x}^*\|$.

To see this we observe that

$$\|x^*\| \leqslant \|\hat{x}^*\| + \|x^*-\hat{x}^*\|.$$

Therefore, (2.2.6) gives

$\|x^*-\hat{x}^*\| \leqslant p\,\|\hat{x}^*\| + p\,\|x^*-\hat{x}^*\|$ and hence the result.

Suppose we have a sequence of the approximate equations giving a sequence of approximate solutions. Then the operators $\hat{U}$, $\hat{F}$, P and the constants α, α_1, α_2, q, p, ε all depend on n.

Theorem 2.2.3

Let the following conditions be satisfied for each n.

i) The operator $\hat{F}$ satisfies the hypothesis of corollary 2.2.1.

ii) $\hat{F}^{-1}$ exists and is linear.

iii) The inequalities of definitions 2.1.1, 2.1.2 and 2.1.3 are satisfied.

Now if

(2.2.11)a $$\lim_{n\to\infty} \alpha^{(n)} = 0,$$

(2.2.11)b $$\lim_{n\to\infty} \alpha_1^{(n)}\|P_n\| = 0,$$

and

(2.2.11)c $$\lim_{n\to\infty} \alpha_2^{(n)}\|P_n\| = 0,$$

then the sequence of the approximate equations are soluble and the the sequence of the approximate solutions converges to the exact solution:

$$\lim_{n\to\infty} \| x^* - \hat{x}_n^* \| = 0.$$

<u>Proof:</u>

Since $P_n = P_n^2$, $\| P_n \| \geqslant 1$ for each n. From theorem 2.2.1 for sufficiently large n, we can find q_n such that $q_n < \frac{1}{2}$ and equation (2.1.2) is soluble for each n. From the conditions of corollary 2.2.1 we have $\| \hat{F}^{-1} \| \leqslant \frac{M}{1-q_n} < 2M.$

Hence, $\{ \hat{F}_n^{-1} \}$ is uniformly bounded.

From (2.2.7) and (2.2.6) we have

$$\begin{aligned}\|x^* - \hat{x}_n^*\| &\leqslant 4|\lambda|M\|x^*\|\,\alpha^{(n)} + |\lambda|(1+\|\hat{F}_n^{-1}P_nF\|)\|x^*\|\,\alpha_1^{(n)} \\ &\quad + \|F\|(1+\|\hat{F}_n^{-1}P_nF\|)\|x^*\|\,\alpha_2^{(n)} \\ &\leqslant 4|\lambda|M\|x^*\|\,\alpha^{(n)} + |\lambda|(\|P_n\| + \|\hat{F}_n^{-1}\|\,\|P_n\|\,\|F\|)\,\alpha_1^{(n)}\|x^*\| \\ &\quad + \|F\|(\|P_n\| + \|\hat{F}_n^{-1}\|\,\|P_n\|\,\|F\|)\,\alpha_2^{(n)}\|x^*\| \\ &\leqslant 4|\lambda|M\alpha^{(n)}\|x^*\| + |\lambda|(1+2M\|F\|)\,\alpha_1^{(n)}\|P_n\|\,\|x^*\| \\ &\quad + \|F\|(1+2M\|F\|)\,\alpha_2^{(n)}\|P_n\|\,\|x^*\|.\end{aligned}$$

Therefore,

$$\text{(2.2.12)} \qquad \|x^* - \hat{x}_n^*\| \leqslant B_0\,\alpha^{(n)} + B_1\,\alpha_1^{(n)}\|P_n\| + B_2\,\alpha_2^{(n)}\|P_n\|,$$

where

$$B_0 = 4|\lambda|M\|x^*\|$$

$$B_1 = |\lambda|(1+2M\|F\|)\|x^*\|$$

$$B_2 = \|F\|(1+2M\|F\|)\|x^*\|.$$

(2.2.11)a, (2.2.11)b and (2.2.11)c give

$$\lim_{n\to\infty} \| x^* - \hat{x}^*_n \| = 0$$

Remark 2.2.2

If we use $\|\hat{F}_n^{-1}\| < 2M$ and $1+\varepsilon_n \leqslant 2$ in (2.2.10) instead of (2.2.7) we get

$$\|x^*-\hat{x}^*_n\| \leqslant 4M|\lambda|\, \alpha^{(n)} \|x^*\| + \varepsilon_n \left(\|P_n\| + \|\hat{F}_n^{-1}\|\, \|P_n\|\, \|F\|\right) \|x^*\|$$

$$= 4M|\lambda|\,\|x^*\|\, \alpha^{(n)} + \|x^*\|\left(1+2M\|F\|\right) \varepsilon_n \|P_n\| .$$

Hence,

(2.2.13) $\quad \|x^*-\hat{x}^*_n\| \leqslant B_0' \alpha^{(n)} + B_1' \varepsilon_n \|P_n\|$

where $B_0'=4M\,|\lambda|\,\|x^*\|$ and $B_1' = \|x^*\|(1+2M\,\|F\|)$.

Moreover, conditions (2.2.11)a, (2.2. 11)b and (2.2.11)c become

(2.2.14) $\quad \lim_{n\to\infty} \alpha^{(n)} = 0 \quad$ and $\lim_{n\to\infty} \varepsilon_n \|P_n\| = 0$.

Theorem 2.2.4

Let $\hat{F}$ have a linear inverse and let the inequalities of definitions 2.1.1 and 2.1.2 be satisfied. Let also the following condition hold:

(2.2.15) $\quad r=|\lambda|\,\alpha(1+|\lambda|\,\alpha_1)\,\|\hat{F}^{-1}\| + |\lambda|\,\alpha_1\,(1+\|\hat{F}^{-1}PF\|) < 1$.

Then the operator F has a linear inverse such that

$$\|F^{-1}\| \leqslant \frac{1+\|\hat{F}^{-1}P\| + |\lambda|\,\alpha\,\|\hat{F}^{-1}\| + \|\hat{F}^{-1}PF\|}{1-r}.$$

Proof:

Let x^* be an element in X. Then x^* is a solution of $F(x) = F(x^*)$. From definition 2.1.2 we can find $\hat{x}\in\hat{X}$ such that

$$\|U(x^*)-\hat{x}\| \leqslant \alpha_1 \|x^*\| .$$

From (2.1.1) we have

$$x^* = \lambda U(x^*) + F(x^*).$$

Therefore,

$$\begin{aligned}\| x^* - \lambda \hat{x} \| &= \| \lambda U(x^*) - \lambda \hat{x} + F(x^*) \| \\ &\leqslant |\lambda| \, \| U(x^*) - \hat{x} \| + \| F(x^*) \| \\ &\leqslant |\lambda| a_1 \| x^* \| + \| F(x^*) \| \\ &= \left(|\lambda| a_1 + \frac{\| F(x^*) \|}{\| x^* \|} \right) \| x^* \| \\ &= \varepsilon \, \| x^* \| \end{aligned}$$

where ε given by

$$(2.2.16) \qquad \varepsilon = \left(|\lambda| a_1 + \frac{\| F(x^*) \|}{\| x^* \|} \right).$$

Consider the equation $\hat{F}(\hat{x}) = F(x^*)$. Let the approximate equation for this equation be $\hat{F}(\hat{x}) = PF(x^*)$ with a solution $\hat{x}^* = \hat{F}^{-1} PF(x^*)$.

By substituting from (2.2.16) in (2.2.10) we have

$$\begin{aligned}\| x^* - \hat{F}^{-1} PF(x^*) \| \leqslant & \Big[\| \hat{F}^{-1} \| \, |\lambda| a \left(1 + |\lambda| a_1 + \frac{\| F(x^*) \|}{\| x^* \|} \right) \\ & + \left(|\lambda| a_1 + \frac{\| F(x^*) \|}{\| x^* \|} \right) \left(1 + \| \hat{F}^{-1} PF \| \right) \Big] \| x^* \| \\ = & \| \hat{F}^{-1} \| \, |\lambda| a \left(\| x^* \| + |\lambda| a_1 \| x^* \| + \| F(x^*) \| \right) \\ & + \left(|\lambda| a_1 \| x^* \| + \| F(x^*) \| \right) \left(1 + \| \hat{F}^{-1} PF \| \right) \\ = & \left[|\lambda| a \| \hat{F}^{-1} \| (1 + |\lambda| a_1) + |\lambda| a_1 (1 + \| \hat{F}^{-1} PF \|) \right] \| x^* \| \\ & + \left[\| \hat{F}^{-1} \| \, |\lambda| a + 1 + \| \hat{F}^{-1} PF \| \right] \| F(x^*) \| \\ = & \; r \| x^* \| + \left[\| \hat{F}^{-1} \| \, |\lambda| a + 1 + \| \hat{F}^{-1} PF \| \right] \| F(x^*) \| \end{aligned}$$

where r as given by (2.2.15).

But

$$\| x^* \| \leqslant \| x^* - \hat{F}^{-1} PF(x^*) \| + \| \hat{F}^{-1} PF(x^*) \| .$$

Hence,

$$\|x^*\| \leq r\|x^*\| + \left[\|\hat{F}^{-1}\|\,|\lambda|\,\alpha + 1 + \|\hat{F}^{-1}PF\|\right]\|F(x^*)\| + \|\hat{F}^{-1}P\|\,\|F(x^*)\|$$

$$= r\|x^*\| + \left[1 + \|\hat{F}^{-1}P\| + \|\hat{F}^{-1}\|\,|\lambda|\alpha + \|\hat{F}^{-1}PF\|\right]\|F(x^*)\| .$$

This gives

$$\frac{(1-r)\|x^*\|}{1 + \|\hat{F}^{-1}P\| + \|\hat{F}^{-1}\|\,|\lambda|\,\alpha + \|\hat{F}^{-1}PF\|} \leq \|F(x^*)\| .$$

From theorem 1.3.5 with

$$m = \frac{1-r}{1 + \|\hat{F}^{-1}P\| + \|\hat{F}^{-1}\|\,|\lambda|\alpha + \|\hat{F}^{-1}PF\|}$$

we find that F has linear inverse such that

$$\|F^{-1}\| \leq \frac{1 + \|\hat{F}^{-1}P\| + \|\hat{F}^{-1}\|\,|\lambda|\,\alpha + \|\hat{F}^{-1}PF\|}{1-r} .$$

Suppose the inequality of definition 2.1.3 is satisfied only for large constant α_2. In other words, we can not find an element in $\hat{X}$ as a close approximation to the exact solution of (2.1.1). To resolve this difficulty take $x=y+z$ where $x \epsilon X$ and y is the free term of (2.1.1).

From (2.1.1) we have

$$F(z)+F(y) = F(z+y) = x + y - \lambda U(z) - \lambda U(y) = y$$

$$= z - \lambda U(z) + (y - \lambda U(y))$$

$$= y' + y - \lambda U(y) .$$

From (2.1.1) we have

(2.2.17) $\quad F(z) = z - \lambda U(z) = y'$.

Therefore,

$$y' + y + \lambda U(y) = y \text{ and hence } \frac{y'}{\lambda} = U(y) .$$

Consequently from definition 2.1.2, for every $y \epsilon X$ there exists $\frac{\hat{y}}{\lambda} \epsilon \hat{X}$ such that

$$\left\| U(y) - \frac{\hat{y}}{\lambda} \right\| \leqslant \alpha_1 \| y \|$$

or

$$\| \lambda U(y) - \hat{y} \| \leqslant |\lambda| \alpha_1 \| y \| .$$

Hence we have

$$\| y' - \hat{y} \| \leqslant |\lambda| \alpha_1 \| y \| . \tag{2.2.18}$$

Thus for y' we can find $\hat{y} \epsilon \hat{X}$ such that

$$\| y' - \hat{y} \| \leqslant \alpha_2^* \| y \| , \text{ where } \alpha_2^* = |\lambda| \alpha_1 .$$

This means that the inequality of definition 2.1.3 holds for y' with $\alpha_2^* = |\lambda| \alpha_1$.

If x^* is a solution of (2.1.1) then $x^* = y + z^*$ and z^* is a solution of (2.2.17).

But the approximate equation of (2.2.17) is given by

$$\hat{F}(\hat{z}) = P(y') = \lambda PU(y). \tag{2.2.19}$$

If $\hat{z}^*$ is a solution of (2.2.19), then $\hat{z}^*$ is an approximate solution of (2.2.17). Hence $x' = y + \hat{z}^*$ is an approximate solution of (2.1.1) with error given by

$$\| x^* - x' \| = \| z^* + y - y - \hat{z}^* \| = \| z^* - \hat{z}^* \| .$$

From theorem 2.2.2, $\| z^* - \hat{z}^* \| \leqslant p \| z^* \|$ with p defined by (2.2.7) and $\alpha_2 = |\lambda| \alpha_1$.

Hence,

$$\| x^* - x' \| = \| z^* - \hat{z}^* \| \leqslant p \| z^* \| .$$

We put the conclusion of this discussion in the following theorem.

Theorem 2.2.5

Let x^* be a solution of equation (2.1.1). Assume that $\hat{F}^{-1}$ exists and is linear and the inequalities of definitions 2.1.1 and 2.1.2 hold. Then,

$$(2.2.20) \qquad \| x^* - x' \| \leqslant p' \| x^* \|$$

where p' is given by

$$(2.2.21) \qquad p' = |\lambda| \left[a_1 (1 + \| \hat{F}^{-1} PF \|) + |\lambda| a \| \hat{F}^{-1} \| \left(\| U \| + a_1 \right) \right].$$

Proof:

From (2.1.1) we have

$$F(x^*) = x^* - \lambda U(x^*) = y.$$

Then

$$y + z^* - \lambda U(z^*) - \lambda U(y) = y \quad \text{if } x^* = y + z^*.$$

This gives

$$\frac{z^*}{\lambda} = U(x^*).$$

Hence, from definition 2.1.2 there exists $\frac{\hat{z}}{\lambda}$ such that

$$\left\| U(x^*) - \frac{\hat{z}}{\lambda} \right\| \leqslant a_1 \| x^* \| .$$

Therefore,

$$\| z^* - \hat{z} \| \leqslant |\lambda| \, a_1 \| x^* \|$$

Let $\hat{z}_o = \hat{F}^{-1} PF(\hat{z})$ and let z^* and $\hat{z}^*$ be solutions of (2.2.17) and (2.2.19) respectively. Then

$$F(z^*) = y' = \lambda U(y)$$

and

$$\hat{F}(\hat{z}^*) = P(y') = PF(z^*)$$

which gives

$$\hat{z}^* = \hat{F}^{-1} PF(z^*).$$

Hence,

$$||\hat{z}^*-\hat{z}_0|| = ||\hat{F}^{-1}PF(z^*)-\hat{F}^{-1}PF(\hat{z})|| = ||\hat{F}^{-1}PF(z^*-\hat{z})||$$
$$\leqslant ||\hat{F}^{-1}PF|| \; ||z^*-\hat{z}|| \leqslant |\lambda|\, a_1 ||\hat{F}^{-1}PF|| \; ||x^*|| .$$

Now

$$||z-\hat{z}_0|| = ||\hat{F}^{-1}F(\hat{z}) - \hat{F}^{-1}PF(\hat{z})||$$
$$\leqslant ||\hat{F}^{-1}|| \; ||\hat{F}(\hat{z}) - PF(\hat{z})|| \leqslant |\lambda|\, a \, ||\hat{F}^{-1}|| \cdot ||\hat{z}||$$
$$\leqslant |\lambda|\, a ||\hat{F}^{-1}|| \left[||z^*|| + ||\hat{z} - z^*|| \right]$$
$$\leqslant |\lambda|\, a ||\hat{F}^{-1}|| \left[|\lambda| \; ||U(x^*)|| + |\lambda| a_1 ||x^*|| \right]$$
$$\leqslant |\lambda|\, a ||\hat{F}^{-1}|| \left[|\lambda| \; ||U|| \; ||x^*|| + |\lambda| \; a_1 \; ||x^*|| \right]$$
$$= |\lambda|^2 a \, ||\hat{F}^{-1}|| \cdot \left(||U|| + a_1 \right) ||x^*|| .$$

Therefore,

$$||x^* - x'|| = || z^* + y - \hat{z}^* - y|| = ||z^* - \hat{z}^*||$$
$$\leqslant || z^* - \hat{z}|| + ||\hat{z} - \hat{z}_0|| + ||\hat{z}_0 - \hat{z}^*||$$
$$\leqslant |\lambda| a_1 || x^*|| + |\lambda|^2 a \, ||\hat{F}^{-1}|| \left(||U|| + a_1 \right) || x^*||$$
$$+ |\lambda| \; a_1 \, ||\hat{F}^{-1}PF|| \; ||x^*||$$
$$= |\lambda| \left[a_1 (||\hat{F}^{-1}PF|| + 1) + |\lambda|\, a \, ||\hat{F}^{-1}|| (||U|| + a_1) \right] ||x^*||$$
$$= P' \, ||x^*||$$

where p' is defined by (2.2.21).

Remark 2.2.3

If the approximate equation (2.1.2) is constructed from (2.1.1) by replacing $\hat{U}$ by PU, then we have to replace $\hat{F}$ by PF to get

$$||PU(\hat{x}) - \hat{U}(\hat{x})|| = 0 .$$

In other words, the inequality of definition 2.1.1 is satisfied with $a = 0$. Therefore the statements of theorems 2.2.1 through 2.2.5 become simpler.

Theorem 2.2.6

Let the following conditions be satisfied:

i) X is complete space,

ii) $P_n \longrightarrow I$ on X,

iii) U is completely continuous.

Then the corresponding sequences $\{\alpha_1^{(n)}\}$, $\{\alpha_2^{(n)}\}$ that appear in definitions 2.1.2 and 2.1.3 respectively, can be chosen in such a way that each converges to zero.

Proof:

Consider the unit sphere E of the space X.

Since U is completely continuois, U(E) is compact.

Thus from theorem 1.3.6

$P_n \longrightarrow I$ uniformly on U(E).

Let $\alpha_1^{(n)}$ be defined by

(2.2.22) $$\sup_{z \in U(E)} \| Pn(z) - z \| = \alpha_1^{(n)} .$$

Hence $\alpha_1^{(n)} \longrightarrow 0$. For each x we choose $P_n U(x)$ such that the following inequality is satisfied:

$$\| U(x) - P_n U(x) \| \leqslant \alpha_1^{(n)} \| x \| .$$

Since $P_n U(x) \in \hat{X}_n$ with $\alpha_1^{(n)}$, conditions of definition 2.1.2 hold. Let $\alpha_2^{(n)} = \dfrac{\| y - P_n(y) \|}{\| y \|}$. Then $\alpha_2^{(n)} \longrightarrow 0$ and

(2.2.23) $$\| y - Pn(y) \| = \alpha_2^{(n)} \| y \| .$$

Since $P_n(y) \in \hat{X}_n$, $\alpha_2^{(n)}$ satisfies the inequality of definition 2.1.3. Thus by choosing $\alpha_1^{(n)}$, $\alpha_2^{(n)}$ as above we see that theorem holds.

<u>Remark 2.2.4</u>

If the spaces X and $\hat{X}$ coincide then the inequalities of definitions 2.1.1, 2.1.2 and 2.1.3 are satisfied with $\| U - \hat{U} \| \leq a$ and $a_1 = a_2 = 0$.

CHAPTER III

APPROXIMATION THEORY IN DIFFERENT SPACE FROM THE ORIGINAL SPACE

3.1 Transformation of the Approximate Equation

Let $\hat{X} \subset X$ be a complete subspace of X.

Let A_o be a one-to-one linear operator mapping $\hat{X}$ onto the complete space W. Since A_o is 1-1 and onto, it is guaranteed that A_o^{-1} exists and is linear. Let A be a linear extension of A_o from X onto W such that A_o and A coincide on $\hat{X}$.

Let P be a projection of X onto $\hat{X}$ with $P^2 = P$. We note that P does not change the elements of $\hat{X}$. Let us define A by the following relation:

(3.1.1) $\qquad A = A_oP$

so that

(3.1.2) $\qquad P = A_o^{-1}A.$

Since A_o is one-to-one and onto, for each $\hat{x} \epsilon \hat{X}$ there exists an element $\bar{x} \epsilon W$ such that

(3.1.3) $\qquad A_o(\hat{x}) = \bar{x}$ and $\hat{x} = A_o^{-1}(\bar{x})$ for $\hat{x} \epsilon \hat{X}$, $\bar{x} \epsilon W$.

From (2.1.2), (3.1.2) and (3.1.3) we have:

$$\hat{F}\, A_o^{-1}(\bar{x}) = A_o^{-1}(\bar{x}) - \lambda \hat{U} A_o^{-1}(\bar{x}) = A_o^{-1}A(y).$$

Applying A_o to both sides, we get

$$A_o\hat{F}A_o^{-1}(\bar{x}) = \bar{x} - \lambda A_o\hat{U}A_o^{-1}(\bar{x}) = A(y).$$

If

(3.1.4) $\bar{F} = A_o \hat{F} A_o^{-1}$, $\bar{U} = A_o \hat{U} A_o^{-1}$ and $A(y) = \bar{y}$,

then the above reduces to

(3.1.5) $\bar{F}(\bar{x}) = \bar{x} - \lambda \bar{U}(\bar{x}) = \bar{y}$.

Thus the approximate equation (2.1.2), in the subspace $\hat{X}$ is transformed to an equivalent equation (3.1.5) in the space W.

Since the definitions 2.1.2 and 2.1.3 do not contain the operators $\hat{U}$ and P they are unchanged in the new situation. But form definition 2.1.1 and equation (3.1.4) we respectively have

$$\| PU(\hat{x}) - \hat{U}(\hat{x}) \| \leqslant a \; \| \hat{x} \|$$

and $\hat{U} = A_o^{-1} \bar{U} A_o$.

Therefore,

(3.1.6) $\| A_o^{-1} AU(\hat{x}) - A_o^{-1} \bar{U} A_o(\bar{x}) \| \leqslant a \; \| \hat{x} \|$.

Let the inequality (3.1.6) be satisfied if and only if the following inequality

$$\| AU(\hat{x}) - \bar{U} A_o(\hat{x}) \| \leqslant \bar{a} \; \| \hat{x} \|$$

is satisfied. Then we have to take a such that

(3.1.7) $a = \bar{a} \| A_o^{-1} \|$.

Thus definition 2.1.1 can be transformed to the following definition:

<u>Definition 3.1.1</u>

If for every $\hat{x} \epsilon \hat{X}$, the inequality

(3.1.8) $\| AU(\hat{x}) - \bar{U} A_o(\hat{x}) \| \leqslant \bar{a} \; \| \hat{x} \|$

with $\bar{a} = \dfrac{a}{\| A_o^{-1} \|}$

is satisfied then the operators AU and $\bar{U}A_o$ are said to be neighboring operators.

Theorem 3.1.1

The inequality (3.1.8) is equivalent to

(3.1.9) $$\|\bar{F}A_o(\hat{x}) - AF(\hat{x})\| \leqslant \bar{\alpha}\,|\lambda|\,\|\hat{x}\|.$$

Proof:

Applying A on (2.1.1) we get

$$AF(\hat{x}) = \hat{x} - \lambda AU(\hat{x}).$$

From (3.1.4) we have

$$\bar{U}A_o = A_oU.$$

Applying A_o to (2.1.2) and A_o to (3.1.4) and then using the results together, we get

$$\bar{F}A_o(\hat{x}) = A_o(\hat{x}) - \lambda\,\bar{U}A_o(\hat{x}).$$

Consequently,

$$\begin{aligned}\|\bar{F}A_o(\hat{x}) - AF(\hat{x})\| &= \|A_o(\hat{x}) - \lambda\bar{U}A_o(\hat{x}) - A(\hat{x}) + \lambda AU(\hat{x})\| \\ &\leqslant |\lambda|\,\|AU(\hat{x}) - \bar{U}A_o(\hat{x})\| \\ &\leqslant |\lambda|\,\bar{\alpha}\,\|x\|.\end{aligned}$$

Conversely, let $\|\bar{F}A_o(\hat{x}) - AF(\hat{x})\| \leqslant |\lambda|\,\bar{\alpha}\,\|\hat{x}\|$.

Then,

$$\begin{aligned}\|AU(\hat{x}) - \bar{U}A_o(\hat{x})\| &= \left\|\frac{AF(\hat{x}) + A(\hat{x}) - A(\hat{x}) - \bar{F}A_o(\hat{x})}{\lambda}\right\| \\ &= \frac{1}{|\lambda|}\|AF(\hat{x}) - \bar{F}A_o(\hat{x})\| \\ &\leqslant \frac{1}{|\lambda|}\cdot|\lambda|\,\bar{\alpha}\,\|\hat{x}\| = \bar{\alpha}\,\|\hat{x}\|.\end{aligned}$$

Remark 3.1.1

If we take $A_o(\hat{x}) = \bar{x}$ then inequalities (3.1.8) and (3.1.9) respectively become:

(3.1.10) $$\|AUA_o^{-1}(\bar{x}) - \bar{U}(\bar{x})\| \leqslant \bar{\alpha}\,\|A_o^{-1}(\bar{x})\| \leqslant \alpha\,\|\bar{x}\|,$$

and

(3.1.11) $$\|\bar{F}(\bar{x}) - AF(\hat{x})\| \leq \bar{\alpha}\,|\lambda|\,\|\hat{x}\| \leq \alpha|\lambda|\,\|\bar{x}\|.$$

3.2 <u>The Approximation Theorems</u>

<u>Theorem 3.2.1</u>

Let the conditions of definitions 3.1.1 and 2.1.2 be satisfied. Assume that the operator F has a linear inverse and

(3.2.1) $$\bar{q} = |\lambda|\left[\bar{\alpha}(1+|\lambda|\alpha_1)\,\|A_o^{-1}\| + \alpha_1\,\|A_o^{-1}AF\|\right]\,\|F^{-1}\| < 1.$$

Then the equation

(3.2.2) $$\bar{F}(\bar{x}) = \bar{y}$$

has a solution $\bar{x}^*$ for every $\bar{y} \in W$ with

(3.2.3) $$\|\bar{x}^*\| \leq \frac{\bar{M}}{1-\bar{q}}\,\|\bar{y}\|$$

and

(3.2.4) $$\bar{M} = (1+|\lambda|\,\alpha_1)\,\|A_o\|\,\|A_o^{-1}\|\,\|F^{-1}\|.$$

<u>Proof:</u>

From theorem 2.2.1 the equation

$$\hat{F}(\hat{x}) = A_o^{-1}(\bar{y})$$

has a solution $\hat{x}^*$ that satisfies (2.2.3), (2.2.1) and (2.2.4) for every $A_o^{-1}(\bar{y}) \in \hat{X}$. Let $\hat{x}^* = A_o^{-1}(\bar{x}^*)$. Then,

$$\hat{F}A_o^{-1}(\bar{x}^*) = A^{-1}(\bar{y}) \text{ this implies } A_o\hat{F}A_o^{-1}(\bar{x}^*) = \bar{y}.$$

From (3.1.4) we have

$$\bar{F}(\bar{x}^*) = \bar{y}.$$

Thus (3.2.2) has a solution for every $\bar{y} \in W$.

Now we show that $\bar{x}^*$ satisfies (3.2.3) with $\bar{M}$ and $\bar{q}$ given by (3.2.4) and (3.2.1) respectively.

$$\|\bar{x}^*\| = \|A_o(\hat{x}^*)\| \leqslant \|A_o\| \ \|\hat{x}^*\| \leqslant \frac{M}{1-q} \|A_o\| \ \|\hat{y}\|$$
$$= \frac{M}{1-q}\|A_o\| \ \|A_o^{-1}(\bar{y})\| \leqslant \frac{M}{1-q}\|A_o\| \ \|A_o^{-1}\| \ \|\bar{y}\| .$$

To replace q by its equivalent $\bar{q}$ in the new situation, we have to replace α and P by $\bar{\alpha}\|A_o^{-1}\|$ and $A_o^{-1}A$ respectively in (2.2.1) to obtain

$$\|\bar{x}^*\| \leqslant \frac{\bar{M}}{1-\bar{q}} \|\bar{y}\| ,$$

with

$$\bar{M} = M\|A_o\| \ \|A_o^{-1}\| = (1+ |\lambda| \alpha_1)\|A_o\| \ \|A_o^{-1}\| \ \|F^{-1}\|$$

and

$$\bar{q} = |\lambda| \left[\bar{\alpha}\|A_o^{-1}\|(1+ |\lambda| \alpha_1) + \alpha_1 \|A_o^{-1}AF\| \right] \|F^{-1}\| < 1 .$$

<u>Corollary 3.2.1</u>

Let the operator $\bar{U}$ be such that the existence of a solution of (3.2.2) for every $\bar{y} \in W$ implies its uniqueness. Then given the conditions of theorem 3.2.1, $\bar{F}$ has an inverse and

$$\|\bar{F}^{-1}\| \leqslant \frac{\bar{M}}{1-\bar{q}} . \tag{3.2.5}$$

<u>Proof:</u>

The existence and linearity may be easily seen as in the proof of Corollary 2.2.1.

We show that $\|\bar{F}^{-1}\| \leqslant \frac{\bar{M}}{1-\bar{q}}$. From theorem 3.2.1 we have

$$\|\bar{x}^*\| \leqslant \frac{\bar{M}}{1-\bar{q}} \|\bar{y}\| .$$

Since $\bar{x}^* = \bar{F}^{-1}(\bar{y})$, we have $\|\bar{F}^{-1}(\bar{y})\| \leqslant \frac{\bar{M}}{1-\bar{q}} \|\bar{y}\|$. Hence

$$\|\bar{F}^{-1}\| \leqslant \sup_{\|\bar{y}\|=1} \|\bar{F}^{-1}(\bar{y})\| \leqslant \frac{\bar{M}}{1-\bar{q}} \sup_{\|\bar{y}\|=1} \|\bar{y}\| = \frac{\bar{M}}{1-\bar{q}} .$$

Theorem 3.2.2

Let the inequalities of definitions 3.1.1, 2.1.2 and 2.1.3 be satisfied, and let $\bar{F}^{-1}$ exist. Assume that (2.1.1) and (3.1.5) have solutions x^* and $\bar{x}^*$ respectively. Then

$$\|x^* - A_o^{-1}(\bar{x}^*)\| \leqslant \bar{p}\,\|x^*\|, \tag{3.2.6}$$

where $\bar{p}$ is given by

$$\bar{p} = (1+\varepsilon)\ |\lambda|\ \bar{\alpha}\,\|A_o^{-1}\bar{F}^{-1}\| + \varepsilon(1 + \|A_o^{-1}\bar{F}^{-1}AF\|). \tag{3.2.7}$$

Moreover, ε satisfies

$$\varepsilon \leqslant \alpha_1 |\lambda| + \alpha_2 \|F\|. \tag{3.2.8}$$

Proof:

We show that there exists $A_o^{-1}(\bar{x}) = \hat{x} \in \hat{X}$ such that

$$\|x^* - A_o^{-1}(\bar{x})\| \leqslant \varepsilon\ \|x^*\| \tag{3.2.9}$$

and ε satisfied (3.2.8).

There exist $\hat{z}, \hat{y} \in \hat{X}$ such that

$$\|U(x^*) - \hat{z}\| \leqslant \alpha_1 \|x^*\|$$

and

$$\|y-\hat{y}\| \leqslant \alpha_2\ \|y\| \leqslant \alpha_2\ \|F\|\ \|x^*\| .$$

Use of x^* in (2.1.1) gives $x^* = \lambda U(x^*) + y$.

Now if we let $\hat{x} = \lambda\hat{z}+\hat{y} = A_o^{-1}(\bar{x})$ then

$$\begin{aligned} \|x^* - A_o^{-1}(\bar{x})\| &\leqslant \|\lambda U(x^*) + y - \lambda\hat{z} - \hat{y}\| \\ &\leqslant |\lambda|\,\|U(x^*) - \hat{z}\| + \|y-\hat{y}\| \\ &\leqslant |\lambda|\ \alpha_1 \|x^*\| + \alpha_2\,\|F\|\,\|x^*\|. \end{aligned}$$

Thus we have

$$\|x^* - A_o^{-1}(\bar{x})\| \leqslant \varepsilon\ \|x^*\| \tag{3.2.10}$$

where

$$\varepsilon = \min(1, |\lambda|\ \alpha_1 + \alpha_2 \|F\|).$$

Hence, if $\bar{x}_o = \bar{F}^{-1} AF(\hat{x})$ then

$$\| x^* - A_o^{-1}(\bar{x}^*) \| = \| x^* - A_o^{-1}(\bar{x}) + A_o^{-1}(\bar{x}) - A_o^{-1}(\bar{x}) + A_o^{-1}(\bar{x}_o) - A_o^{-1}(\bar{x}^*) \|$$

$$= \| x^* - A_o^{-1}(\bar{x}) + A_o^{-1}(\bar{x}) - A_o^{-1}\bar{F}^{-1} AF(\hat{x}) + A_o^{-1}\bar{F}^{-1} AF(\hat{x}) - A_o^{-1}(\bar{x})^* \|$$

$$\leqslant \varepsilon \|x^*\| + \| A_o^{-1}(\bar{x}) - A_o^{-1}\bar{F}^{-1} AF(\hat{x}) \|$$

$$+ \| A^{-1}\bar{F}^{-1} AF(\hat{x}) - A_o^{-1}(\bar{x}^*) \|$$

$$\leqslant \varepsilon \|x^*\| + \| A_o^{-1}\bar{F}^{-1}\bar{F}(\bar{x}) - A_o^{-1}\bar{F}^{-1} AF(\hat{x}) \|$$

$$+ \| A_o^{-1}\bar{F}^{-1} AF(\hat{x}) - A_o^{-1}\bar{F}^{-1} AF(x^*) \| ,$$

since $\bar{x}^* = \bar{F}^{-1} AF(x^*)$.

$$\leqslant \varepsilon \|x^*\| + \| A_o^{-1}\bar{F}^{-1} \| \ \| \bar{F}(\bar{x}) - AF(\hat{x}) \|$$

$$+ \| A_o^{-1}\bar{F}^{-1} AF \| \ \| A_o^{-1}(\bar{x}) - x^* \|$$

$$\leqslant \varepsilon \| x^* \| + |\lambda| \ \| A_o^{-1}\bar{F}^{-1} \| \|\bar{\alpha}\| \hat{x} \| + \| A_o^{-1}\bar{F}^{-1} AF \| \varepsilon \| x^* \| .$$

But $\| \hat{x} \| = \| A_o^{-1}(\bar{x}) \| \leqslant \|x^*\| + \| x^* - A_o^{-1}(\bar{x}) \|$

$$\leqslant \|x^*\| + \varepsilon \|x^*\| = (1+\varepsilon) \|x^*\| .$$

Hence,

$$\| x^* - A_o^{-1}(\bar{x}^*) \| \leqslant \varepsilon \|x^*\| + |\lambda| \bar{\alpha} \| A_o^{-1}\bar{F}^{-1} \| (1+\varepsilon) \|x^*\|$$

$$+ \| A_o^{-1}\bar{F}^{-1} AF \| \ \| x^* \| .$$

This gives

$$\| x^* - A_o^{-1}(\bar{x}^*) \| \leqslant \bar{p} \|x^*\|,$$

where p satisfies (3.2.7) .

Remark 3.2.1

Since $1+\varepsilon \leqslant 2$ and $\|A_o^{-1}\bar{F}^{-1}\| = \|A_o^{-1}\bar{F}^{-1}A_oA_o^{-1}\| \leqslant \|A_o^{-1}\|\,\|A_o^{-1}\bar{F}^{-1}A_o\|$, from (3.2.7) we have

$$\|x^* - A_o^{-1}(x^*)\| \leqslant \Big[2|\lambda|\bar{a}\,\|A_o^{-1}\|\,\|A_o^{-1}\bar{F}^{-1}A_o\|$$

$$+\varepsilon(1+\|A_o^{-1}\bar{F}^{-1}AF\|)\Big]\|x^*\|.$$

Therefore, (3.2.6) is satisfied with $\bar{p}$ given by

(3.2.11) $\bar{p} = 2|\lambda|\bar{a}\|A_o^{-1}\|\,\|A_o^{-1}\bar{F}^{-1}A_o\| + \varepsilon(1+\|A_o^{-1}\bar{F}^{-1}AF\|)$.

Theorem 3.2.3

Let the following conditions be satisfied for each n.

1) The operator $\bar{F}_n$ satisfies the condition of corollary 3.2.1.

2) F^{-1} exists and is linear.

3) Inequalities of definitions 3.1.1, 2.1.2 and 2.1.3 are satisfied.

Moreover, let the following hold.

(3.2.12)a $\quad \lim\limits_{n\to\infty} \bar{a}^{(n)}\|A_{on}^{-1}\| = o$

(3.2.12)b $\quad \lim\limits_{n\to\infty} a_1^{(n)}\|A_{on}^{-1}A_n\| = o$

(3.2.12)c $\quad \lim\limits_{n\to\infty} a_2^{(n)}\|A_{on}^{-1}A_n\| = o$

Then the approximate equation (3.1.5) is soluble for sufficiently large n and the sequence of the approximate solutions converges to the exact solution

i.e., $\lim\limits_{n\to\infty}\|x^* - A_{on}^{-1}(\bar{x}_n^*)\| = o$.

Proof:

Since A_{on} and A_{on}^{-1} are 1-1 correspondances, $\|A_{on}\| \neq o$ and $\|A_{on}^{-1}\| \neq o$.

The operators $A_{on}^{-1}A_n : X \longrightarrow \hat{X}$ and $A_{on}^{-1}A_{on} : \hat{X} \longrightarrow \hat{X}$ are linear and onto. Since A_n is extension of A_{on} defined on $\hat{X}$ to X, we have

$$\left\{ \|A_{on}^{-1}A_{on}(\hat{x})\| \;/\; \|\hat{x}\| \leqslant 1 \;, \hat{x}\in\hat{X} \right\} \subseteq \left\{ \|A_{on}^{-1}A_n(x)\| \;/\; \|x\| \leqslant 1 \;, x\in X \right\}.$$

Therefore,

$$\sup_{\|\hat{x}\|\leqslant 1} \left\{ \|A_{on}^{-1}A_{on}(x)\| / \|\hat{x}\| \leqslant 1 , \hat{x}\in\hat{X} \right\} \leqslant \sup_{\|x\|\leqslant 1} \left\{ \|A_{on}^{-1}A_n(x)\| / \|x\| \leqslant 1 , x\in X \right\}$$

In the other words,

$$1 = \|A_{on}^{-1}A_{on}\| \leqslant \|A_{on}^{-1}A_n\| \text{ and hence } \|A_{on}^{-1}A_n\| \neq o.$$

Then from (3.2.12) it follows that $\lim\limits_{n\to\infty} \bar{\alpha}^{(n)} = o$,

$$\lim_{n\to\infty} \alpha_1^{(n)} = o \text{ and } \lim_{n\to\infty} \alpha_2^{(n)} = o.$$

From condition (3) and (3.2.1), for sufficiently large n we have $\bar{q}^{(n)} < \frac{1}{2}$. From the corollary 3.1.1 we get

$$\|\bar{F}_n^{-1}\| \leqslant \frac{\bar{M}}{1-\bar{q}^{(n)}} < 2\bar{M}. \quad \text{Since}$$

$$\varepsilon^{(n)} \leqslant (|\lambda|\, \alpha_1^{(n)} + \alpha_2^{(n)} \|F\|) \text{ and } \|\bar{F}_n^{-1}\| \leqslant 2\bar{M},\ 1 + \varepsilon^{(n)} \leqslant 2,$$

from (3.2.7) and (3.2.6) we have

$$\begin{aligned} \|x^* - A_{on}^{-1}(x_n^*)\| &\leqslant \Big[2\, |\lambda|\, \|A_{on}^{-1}\|\, \|\bar{F}_n^{-1}\|\, \bar{\alpha}^{(n)} \\ &\quad + (|\lambda|\alpha_1^{(n)} + \alpha_2^{(n)}\|F\|)(1 + \|A_{on}^{-1}\bar{F}_n^{-1}A_nF\|) \Big] \|x^*\| \\ &\leqslant 4\bar{M}\, \|x^*\|\, \bar{\alpha}^{(n)}\, \|A_{on}^{-1}\| \\ &\quad + (|\lambda|\alpha_1^{(n)} + \alpha_2^{(n)})\, \|F\|\,)(1 + \|\hat{F}_n^{-1}A_{on}^{-1}A_nF\|)\, \|x^*\|. \end{aligned}$$

Since $1 \leqslant \|A_o^{-1}A_n\| \leqslant \|A_{on}^{-1}\|\, \|A_n\|$, from the proof of theorem 2.2.3 we have

$$\|\hat{F}_n^{-1}\| \leqslant 2M \leqslant 2M\, \|A_{on}^{-1}\|\, \|A_n\| = 2\bar{M}.$$

Therefore,

$$\| x^* - A_{on}^{-1}(\bar{x}_n^*)\| \leqslant 4\bar{M}\|x^*\| \bar{a}^{(n)} \|A_{on}^{-1}\|$$

$$+ (|\lambda| a_1^{(n)} + a_2^{(n)} \|F\|)(\|A_{on}^{-1}A_n\| + 2\bar{M}\|F\| \|A_{on}^{-1}A_n\|) \|x^*\|$$

$$= 4\bar{M}\|x^*\| \bar{a}^{(n)} \|A_{on}^{-1}\|$$

$$+ (|\lambda| a_1^{(n)} + a_2^{(n)}\|F\|)(1 + 2\bar{M}\|F\|) \|A_{on}^{-1}A_n\| \|x^*\|$$

$$= 4\bar{M}\|x^*\| \bar{a}^{(n)} \|A_{on}^{-1}\|$$

$$+ |\lambda|(1+2\bar{M}\|F\|)\|x^*\| a_1^{(n)} \|A_{on}^{-1}A_n\|$$

$$+ \|F\|(1+2\bar{M}\|F\|)\|x^*\| a_2^{(n)} \|A_{on}^{-1}A_n\| .$$

$$= \bar{B}_o \bar{a}^{(n)} \|A_{on}^{-1}\| + \bar{B}_1 a_1^{(n)} \|A_{on}^{-1}A_n\| + \bar{B}_2 a_2^{(n)} \|A_{on}^{-1}A_n\|$$

where $\bar{B}_o$, $\bar{B}_1$ and $\bar{B}_2$ are given by

$$\bar{B}_o = 4\bar{M}\|x^*\| ,$$

$$\bar{B}_1 = |\lambda|(1+2\bar{M}\|F\|)\|x^*\| ,$$

and

$$\bar{B}_2 = \|F\|(1+2\bar{M}\|F\|)\|x^*\| .$$

Theorem 3.2.4

Let $\bar{F}^{-1}$ exist and be linear. Assume that the inequalities of definitions 3.1.1 and 2.1.2 are satisfied. If $\bar{r}$ is given by

$$\bar{r} = |\lambda|\left[\bar{a}\|A_o^{-1}\| (1+|\lambda| a_1)\|A_o^{-1}\bar{F}^{-1}A_o\| + a_1(1+\|A_o\bar{F}^{-1}AF\|)\right] < 1 \tag{3.2.13}$$

then F has inverse and

$$\|F^{-1}\| \leqslant \frac{1+\|A_o^{-1}\bar{F}^{-1}A\| + |\lambda|\bar{a}\|A_o^{-1}\| \|A_o^{-1}\bar{F}^{-1}A_o\| + \|A_o^{-1}\bar{F}^{-1}AF\|}{1-\bar{r}} . \tag{3.2.14}$$

Proof:

Let x^* be an element of X. Then x^* is a solution of the equation

(3.2.15) $\qquad F(x) = F(x^*).$

From definition 2.1.2 we find an element $\hat{x} = A_o^{-1}(\bar{x}) \epsilon X$ with $\bar{x} \epsilon W$ such that $\|U(x^*) - A_o^{-1}(\bar{x})\| \leqslant \alpha_1 \|x^*\|$.

By equation (2.1.1) we have $x^* = \lambda U(x^*) + F(x^*)$ such that

$$\begin{aligned} \| x^* - \lambda A_o^{-1}(\bar{x}) \| &= \| \lambda U(x^*) - \lambda A_o^{-1}(\bar{x}) + F(x^*) \| \\ &\leqslant |\lambda| \, \| U(x^*) - A_o^{-1}(\bar{x}) \| + \| F(x^*) \| \\ &\leqslant |\lambda| \, \alpha_1 \| x^* \| + \| F(x^*) \| \\ &= \left(|\lambda| \alpha_1 + \frac{\|F(x^*)\|}{\|x^*\|} \right) \|x^*\| = \varepsilon \|x^*\| \end{aligned}$$

where $\qquad \varepsilon = \left(|\lambda| \alpha_1 + \frac{\|F(x^*)\|}{\|x^*\|} \right).$

This inequality enables us to apply theorem 3.2.2 with the new value of ε.

The approximate equation $\bar{F}(\bar{x}) = AF(x^*)$ for (3.2.15) has a solution $\bar{x}^* = \bar{F}^{-1} AF(x^*)$. This implies that $A_o^{-1}(\bar{x}^*) = A^{-1} F^{-1} AF(x^*)$. Hence, from (3.2.7) and (3.2.6) we have

$$\begin{aligned} \| x^* - A_o^{-1}(\bar{x}^*) \| &= \| x^* - A_o^{-1} \bar{F}^{-1} AF(x^*) \| \\ &\leqslant \Big[|\lambda| \bar{\alpha} \| A_o^{-1} \bar{F}^{-1} \| \Big(1 + |\lambda| \alpha_1 + \frac{\|F(x^*)\|}{\|x^*\|} \Big) \\ &\quad + \Big(|\lambda| \alpha_1 + \frac{\|F(x^*)\|}{\|x^*\|} \Big) \big(1 + \| A_o^{-1} \bar{F}^{-1} AF \| \big) \Big] \| x^* \| \\ &= |\lambda| \bar{\alpha} \| A_o^{-1} \bar{F}^{-1} \| (1 + |\lambda| \alpha_1) \| x^* \| + |\lambda| \bar{\alpha} \| A_o^{-1} \bar{F}^{-1} \| \, \| F(x^*) \| \end{aligned}$$

$$+|\lambda|\ \alpha_1(1+\|A_0^{-1}\bar{F}^{-1}AF\|)\|x^*\| + (1+\|A_0^{-1}\bar{F}^{-1}AF\|)\ \|F(x^*)\|$$

$$\leqslant |\lambda|\ \Big[(1+|\lambda|\ \alpha_1)\ \bar{\alpha}\ \|A_0^{-1}\|\ \|A_0^{-1}\bar{F}^{-1}A_0\| + \alpha_1(1+\|A_0^{-1}\bar{F}^{-1}AF\|)\Big]\|x^*\|$$

$$+\Big[1+|\lambda|\ \bar{\alpha}\ \|A_0^{-1}\|\ \|A_0^{-1}\bar{F}^{-1}A_0\| + \|A_0^{-1}\bar{F}^{-1}AF\|\Big]\|F(x^*)\|,$$

$$\text{since}\quad \|A_0^{-1}\|\ \|A_0^{-1}\bar{F}^{-1}A_0\| \geqslant \|A_0^{-1}\bar{F}^{-1}\|$$

$$=\bar{r}\ \|x^*\| + \Big[1+|\lambda|\bar{\alpha}\|A_0^{-1}\|\|A_0^{-1}\bar{F}^{-1}A_0\| + \|A_0^{-1}\bar{F}^{-1}AF\|\Big]\ \|F(x^*)\|.$$

But

$$\|x^*\| \leqslant \|x^* - A_0^{-1}\bar{F}^{-1}AF(x^*)\| + \|A_0^{-1}\bar{F}^{-1}AF(x^*)\|$$

$$\leqslant \bar{r}\|x^*\| + \Big[1+|\lambda|\ \bar{\alpha}\ \|A_0^{-1}\|\ \|A_0^{-1}\bar{F}^{-1}A_0\| + \|A_0^{-1}\bar{F}^{-1}AF\|\Big]\ \|F(x^*)\|$$

$$+\ \|A_0^{-1}\bar{F}^{-1}A\|\ \|F(x^*)\|.$$

Consequently,

$$(1-\bar{r})\ \|x^*\| \leqslant \Big[1 + \|A_0^{-1}\bar{F}^{-1}A\| + |\lambda|\bar{\alpha}\ \|A_0^{-1}\|\ \|A_0^{-1}\bar{F}^{-1}A_0\| + \|A_0^{-1}\bar{F}^{-1}AF\|\Big]\|F(x^*)\|.$$

Therefore,

$$\frac{(1-\bar{r})\ \|x^*\|}{1+\|A_0^{-1}\bar{F}^{-1}A\| + |\lambda|\bar{\alpha}\ \|A_0^{-1}\|\ \|A_0^{-1}\bar{F}^{-1}A_0\| + \|A_0^{-1}\bar{F}^{-1}AF\|} \leqslant \|F(x^*)\|.$$

Now by theorem 1.3.5 F^{-1} exists such that

$$\|F^{-1}\| \leqslant \frac{1+\|A_0^{-1}\bar{F}^{-1}A\| + |\lambda|\bar{\alpha}\ \|A_0^{-1}\|\ \|A_0^{-1}\bar{F}^{-1}A_0\| + \|A_0^{-1}\bar{F}^{-1}AF\|}{1-\bar{r}}.$$

CHAPTER IV

APPLICATIONS

4.1 Application to Infinite System of Equations

Consider the following infinite system of equation:

(4.1.1) $$u_j - \lambda \sum_{k=1}^{\infty} a_{jk} u_k = b_j; \; j = 1, 2, 3, \ldots$$

with the assumptions

(4.1.2) $$\sum_{j,k=1}^{\infty} |a_{jk}|^2 < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} |b_j|^2 < \infty .$$

We have to find a solution for system (4.1.1) that satisfies the condition

(4.1.3) $$\sum_{k=1}^{\infty} |u_k|^2 < \infty .$$

One of the methods by which the required solution can be found is the reduction method which consists of replacing the infinite system (4.1.1) with a system of n equations with n unknowns :

(4.1.4) $$u_j - \lambda \sum_{k=1}^{\infty} a_{jk} u_k = b_j \; ; \; j = 1, 2, 3, \ldots, n.$$

The solution of the finite system (4.1.4) is an approximate solution of the infinite system (4.1.1).

In the following, we investigate whether the approximation theory discussed in earlier chapter is applicable here.

If $X = l^2$, $x = \{u_n\}$ and $y = \{b_n\}$, the system (4.1.1) can be written as a single equation:

$$F(x) = x - \lambda U_{(x)} = y \ , \tag{4.1.5}$$

where U is completely continuous and is defined by the matrix of the system

$$z = U(x), \ v_j = \sum_{k=1}^{\infty} a_{jk} u_k \ , \tag{4.1.6}$$

$$x = \{u_n\} \quad \text{and} \quad z = \{v_n\} \ .$$

Let the space W be the finite-dimentional Euclidean space l_n^2. Let $\hat{X}$ be a subpsace of X consisting of vectors whose coordinates beyond nth are zeros.

Let A_o, A_o^{-1} be defined as in chapter 3. Let A be an extension of A_o defined as follows.

For $x = \{u_m\} \in l^2$, let $A(x) = \bar{x} = (u_1, u_2, \ldots, u_n) \in l_n^2$.

Hence,

$$||A|| = \sup_{||x||=1} ||A(x)|| = \sup_{(\sum_{k=1}^{\infty} |u_k|^2)^{\frac{1}{2}}=1} (\sum_{k=1}^{n} |u_k|^2)^{\frac{1}{2}} = 1 \ .$$

Similarly $||A_o|| = ||A_o^{-1}|| = 1$.

By application of approximation theory, the system (4.1.4) can be written as

$$\bar{F}(\bar{x}) = \bar{x} - \lambda \bar{U}(\bar{x}) = A(y) . \tag{4.1.7}$$

Here $\bar{U}$ is a linear operation in W corresponding to the matrix

$$C_n = \begin{bmatrix} a_{11} & a_{12}\cdots & a_{1n} \\ a_{21} & a_{22}\cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{nn} \end{bmatrix}.$$

Now we show that the inequalities of definitions 3.1.1, 2.1.2 and 2.1.3 are satisfied. For any $\hat{x} = (u_1, u_2, \ldots, u_n, o, \ldots)$ we have

$$\bar{z} = AU(\hat{x}) - \bar{U}A_o(\hat{x}), \quad \bar{z} = (v_1, v_2, \ldots, v_n).$$

Consequently,

$$v_j = \sum_{k=1}^{n} a_{jk}u_k - \sum_{k=1}^{n} a_{jk}u_k = o,$$

and the inequality of definition 3.1.1 is satisfied with $\bar{a} = o$.

If $x = \{u_m\} \epsilon\, l^2$ and $\hat{x} = [U(x)]_n = (v_1, v_2, \ldots, v_n, o, \ldots) \epsilon\, \hat{X}$, then

$$U(x) - \hat{x} = \left(\sum_{k=1}^{\infty} a_{1k}u_k, \sum_{k=1}^{\infty} a_{2k}u_k, \ldots, \sum_{k=1}^{\infty} a_{nk}u_k, \ldots \right)$$

$$- (v_1, v_2, \ldots, v_n, o, \ldots)$$

$$= \left(\sum_{k=1}^{\infty} a_{1k}u_k, \sum_{k=1}^{\infty} a_{2k}u_k, \ldots, \sum_{k=1}^{\infty} a_{nk}u_k, \ldots \right)$$

$$- \left(\sum_{k=1}^{\infty} a_{1k}u_k, \sum_{k=1}^{\infty} a_{2k}u_k, \ldots, \sum_{k=1}^{\infty} a_{nk}u_k, o, \ldots \right)$$

$$= \left(o, o, \ldots, o, \sum_{k=1}^{\infty} a_{n+1,k}u_k, \sum_{k=1}^{\infty} a_{n+2,k}u_k, \ldots \right).$$

Thus

$$\| U_{(x)} - \hat{x} \| = \left[\sum_{j=n+1}^{\infty} \sum_{k=1}^{\infty} |a_{jk} u_k|^2 \right]^{\frac{1}{2}}$$

$$\leqslant \left[\sum_{j=n+1}^{\infty} \sum_{k=1}^{\infty} |a_{jk}|^2 \sum_{k=1}^{\infty} |u_k|^2 \right]^{\frac{1}{2}}$$

$$= \left[\sum_{j=n+1}^{\infty} \sum_{k=1}^{\infty} |a_{jk}|^2 \right]^{\frac{1}{2}} \cdot \left[\sum_{k=1}^{\infty} |u_k|^2 \right]^{\frac{1}{2}}$$

$$= \alpha_1^{(n)} \| x \|$$

where $\alpha_1^{(n)} = \left[\sum_{j=n+1}^{\infty} \sum_{k=1}^{\infty} |a_{jk}|^2 \right]^{\frac{1}{2}}$.

Obviously,

$$\alpha_1^{(n)} \longrightarrow o \quad \text{as } n \longrightarrow \infty .$$

Thus the inequality of definition 2.1.2 is satisfied with this $\alpha_1^{(n)}$.

If $y = \{b_m\} \epsilon X$ and $\hat{y} = [y]_n = (b_1, b_2, \ldots, b_n, o, \ldots)$, then $y - \hat{y} = (o, o, \ldots, o, b_{n+1}, b_{n+2}, \ldots)$.

Thus

$$\| y - \hat{y} \| = \left[\sum_{j=n+1}^{\infty} |b_j|^2 \right]^{\frac{1}{2}} = \left[\sum_{j=n+1}^{\infty} |b_j|^2 \Big/ \sum_{j=1}^{\infty} |b_j|^2 \right]^{\frac{1}{2}} \| y \| .$$

Hence the inequality of the definition 2.1.3 is satisfied with

$$\alpha_2^{(n)} = \left[\sum_{j=n+1}^{\infty} |b_j|^2 \Big/ \sum_{j=1}^{\infty} |b_j|^2 \right]^{\frac{1}{2}} .$$

Clearly, $\alpha_2^{(n)} \longrightarrow o$ as $n \longrightarrow \infty$. Therefore, the approximation theory in chapter 3 is applicable to this system. Thus from theorem 3.2.3, the system (4.1.4) is soluble (if λ is not characteristic value of the operator U) for sufficiently large n and the sequence of the approximate solutions converges to the exact solution of system (4.1.1). The speed of convergence is given by

$$(4.1.8) \qquad \| x^* - A_o^{-1}\bar{x}_n^* \| \leqslant B_1 \left[\sum_{j=n+1}^{\infty} \sum_{k=1}^{\infty} |a_{jk}|^2 \right]^{\frac{1}{2}} + B_2 \left[\sum_{j=n+1}^{\infty} |b_j|^2 \Big/ \sum_{j=1}^{\infty} |b_j|^2 \right]^{\frac{1}{2}},$$

where x^* is the exact solution of (4.1.1) and $\bar{x}_n^*$ is the solution of (4.1.4).

From (4.1.8) we see that $A_o^{-1}\bar{x}_n^* \longrightarrow x^*$ so that $u_k^{*(n)} \longrightarrow u_k^*$.

Hence, from theorem 3.2.4 we have the following theorem.

<u>Theorem 4.1.1</u>

If the system (4.1.4) has unique solution and

$$\bar{r} = |\lambda| \, \alpha_1 \, \|\bar{F}^{-1}\| \; \| AF \| < 1 ,$$

λ will be a non-characteristic value of system (4.1.1) and

$$\| F^{-1} \| \leqslant \frac{1 + \|\bar{F}^{-1}\| \, (1 + \| AF \|)}{1 - \bar{r}} .$$

<u>Proof:</u>

Put $\|A_o\| = \|A_o^{-1}\| = 1$ and $\bar{\alpha} = o$ in theorem 3.2.4.

4.2 Application to Integral Equations

Consider the following equation:

$$x(s) - \lambda\int_0^1 h(s,t)\,x(t)\,dt = y(s)\ ;\quad t,\ s \in [o,1]. \tag{4.2.1}$$

By numerical integration formula based on the points $t_1, t_2, \ldots, t_n$ we have

$$\int_0^1 x(t)\,dt = \sum_{k=1}^{n} M_k x(t)\ . \tag{4.2.2}$$

Equation (4.2.1) is satisfied at only base points. Equations (4.2.1) and (4.2.2) give:

$$x(t_j) - \lambda \sum_{k=1}^{n} M_k h(t_j, t_k)\,x(t_k) = y(t_j)\ ;\ j=1,2,\ldots,n. \tag{4.2.3}$$

This approximates the solution at $t_1, t_2, \ldots, t_n$.

We now show that the approximate theory can be applied to this system and that the equation (4.2.3) can be considered as an approximate equation of equation (4.2.1).

Let $y(s)$, $h(s,t)$ be continuous periodic functions of period 1 in s and t. Let $X = \hat{C}$ be the space of all periodic functions. Let $W = 1_n^{\infty}$. The system (4.2.3) is regarded to be an equation in the space W. Therefore, system (4.2.3) can be represented by the equation

$$\bar{F}(\bar{x}) = \bar{x} - \lambda\,\bar{U}(\bar{x}) = A(y). \tag{4.2.4}$$

For system (4.2.2) we chose

$$M_k = \frac{1}{n} \quad \text{and} \quad t_k = \frac{2k-1}{2n} \; ; \quad k = 1,2,\ldots,n \; .$$

Let the operator $\bar{U}$ be defined by the matrix

$$(4{\cdot}2{\cdot}5) \qquad \begin{bmatrix} \frac{1}{n}h(\frac{1}{2n},\frac{1}{2n}) & \frac{1}{n}h(\frac{1}{2n},\frac{3}{2n}) & \cdots & \frac{1}{n}h(\frac{1}{2n},\frac{2n-1}{2n}) \\ \frac{1}{n}h(\frac{3}{2n},\frac{1}{2n}) & \frac{1}{n}h(\frac{3}{2n},\frac{3}{2n}) & \cdots & \frac{1}{n}h(\frac{3}{2n},\frac{2n-1}{2n}) \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n}h(\frac{2n-1}{2n},\frac{1}{2n}) & \frac{1}{n}h(\frac{2n-1}{2n},\frac{3}{2n}) & \cdots & \frac{1}{n}h(\frac{2n-1}{2n},\frac{2n-1}{2n}) \end{bmatrix} .$$

Suppose $A(x) = (x(t_1),x(t_2),\ldots,x(t_n))$ such that $\|A\| = 1$.

Let $\hat{X}$ be a subspace of X consisting of all continuous periodic functions, linear in every interval of the form $[a_k,a_{k+1}]$ with $a_k = \frac{k}{n}$, $k = o,1,\ldots,n$. Each such function is defined by its value at the points $a_1,a_2,\ldots,a_n$.

Since A coincides with A_o on $\hat{X}$, for all $\hat{x}\epsilon\hat{X}$,

$$A_o(\hat{x}) = A(\hat{x}) = (\hat{x}(t_1),\hat{x}(t_2), \;\ldots,\; \hat{x}(t_n))$$

$$= \bar{x} = (v_1,v_2,\ldots,v_n) \,\epsilon\, W, \text{ and}$$

$$\hat{x}(t_k) = \hat{x}(a_k) \quad \text{since} \quad t_k \,\epsilon\, [a_k,a_{k+1}] \; .$$

This gives $\hat{x}(a_k) = v_k$. Since $\hat{x} = A_o^{-1}(\bar{x})$,

$$\|\bar{A}_o^1\| = \sup_{\|\bar{x}\|=1} \|A_o^{-1}(\bar{x})\| = \sup_{\substack{\max |v_k| = 1 \\ j=1,2,\ldots,n}} \|\hat{x}\| = \sup_{\substack{\max|v_k| = 1 \\ j=1,2,\ldots,n}} \|\hat{x}(a_k)\|$$

$$= \sup_{\substack{\max|v_k| = 1 \\ j=1,2,\ldots,n}} |v_k| = 1$$

Let the modulus of continuity of the function $h(s,t)$ as functions in s and t respectively be given by

$$D_s(\delta) = \sup|h(s+\sigma,t) - h(s,t)| \quad , \quad o \leqslant s,t \leqslant 1, |\sigma| \leqslant \delta ,$$

$$D_t(\delta) = \sup|h(s,t+\sigma) - h(s,t)| \quad , \quad o \leqslant s,t \leqslant 1, |\sigma| \leqslant \delta .$$

Next we find the expression for the error of the formula (4.2.2) for a function of the form $z(t)\hat{x}(t)$, where z is a periodic function such that the molulus of continuity of which does not exceed $D(\delta)$ and $\hat{x} \epsilon \hat{X}$.

Thus

$$(4.2.6) \quad \left| \int_o^1 z(t)\hat{x}(t)dt - \sum_{k=1}^n \frac{1}{n} z(t_k)\hat{x}(t_k) \right|$$

$$= \left| \sum_{k=1}^n \int_{a_k}^{a_{k+1}} z(t)\hat{x}(t)dt - \sum_{k=1}^n \int_{a_k}^{a_{k+1}} z(t_k)\hat{x}(t)dt \right.$$

$$\left. + \sum_{k=1}^n \int_{a_k}^{a_{k+1}} z(t_k)\hat{x}(t)dt - \sum_{k=1}^n \frac{1}{n} z(t_k)\hat{x}(t_k) \right|$$

$$= \left| \sum_{k=1}^n \int_{a_k}^{a_{k+1}} \left[z(t) - z(t_k)\right] \hat{x}(t)dt \right.$$

$$\left. - \frac{1}{n} \sum_{k=1}^n \left[n \int_{a_k}^{a_{k+1}} \hat{x}(t)dt - \hat{x}(t_k)\right] z(t_k) \right| .$$

Since $\hat{x}(t)$ is linear in each of the intervals (a_k, a_{k+1}), we have

$$(4.2.7) \qquad n \int_{a_k}^{a_{k+1}} \hat{x}(t)\,dt = \hat{x}\left(\frac{a_k + a_{k+1}}{2}\right) = \hat{x}(t_k) \quad , \; k = 1,2,\ldots,n-1 .$$

Consequently, the second term of (4.2.6) is zero.

On taking $|\sigma| \leqslant \delta = \frac{1}{2n}$, we get

$$(4.2.8) \qquad \left| \int_o^1 z(t)\hat{x}(t)\,dt - \sum_{k=1}^{n} \frac{1}{n} z(t_k)\hat{x}(t_k) \right|$$

$$= \left| \sum_{k=1}^{n} \int_{a_k}^{a_{k+1}} \left[z(t) - z(t_k) \right] \hat{x}(t)\,dt \right| \leqslant D\left(\frac{1}{2n}\right) \|\hat{x}\| .$$

If $z(t) = h(t_j, t)$ and $z(t_k) = h(t_j, t_k)$ then (4.2.8) gives

$$\|AU(\hat{x}) - \bar{U}A_o(\hat{x})\| = \max_{j=1,2,\ldots,n} \left| \int_o^1 h(t_j, t)\hat{x}(t)\,dt \right.$$

$$\left. - \frac{1}{n}\sum_{k=1}^{n} h(t_j t_k)\hat{x}(t_k) \right|$$

$$\leqslant D_t\left(\frac{1}{2n}\right) \|\hat{x}\| .$$

This shows that the condition of definition 3.1.1 is satisfied with $\bar{\alpha} = D_t(\frac{1}{2n})$.

Let $z \in X = \hat{C}$ be any function whose modulus of continuity does not exceed $D(\delta)$ and let $\hat{z} = A_o^{-1}A(z)$. Then $\hat{z}$ is linear function defined by its values at the points $a_1, a_2, \ldots, a_n$. Since $\hat{z} = A_o^{-1}A(z)$, $A_o\hat{z} = A(z)$ and the corresponding values of the functions z and $\hat{z}$ coincide at the points $a_1, a_2, \ldots a_n$. Thus

if $a_j \leqslant s \leqslant a_{j+1}$, then

$$|z(s) - \hat{z}(s)| = |z(s) - [(a_{j+1}-s)\hat{z}(a_j) + (s-a_j)\hat{z}(a_{j+1})]\, n\,|$$

$$= |z(s) - [(a_{j+1}-s)z(a_j)+(s-a_j)z(a_{j+1})]\, n\,|$$

$$= |z(s).n\,[\,(a_{j+1}-s)+(s-a_j)]$$

$$+ [(a_{j+1}-s)z(a_j)+(s-a_j)z(a_{j+1})]\, n\,|$$

$$= n\,|(a_{j+1}-s)(z(s)-z(a_j))+(s-a_j)(z(s)-z(a_{j+1}))|$$

$$\leqslant n\left[|(a_{j+1}-s)(z(s)-z(a_j))| + |(s-a_j)(z(s)-z(a_{j+1}))|\right]$$

If $|\sigma| = |s-a_j| \leqslant \frac{1}{n} = |a_{j+1}-a_j| = \delta$, then $|z(s)-z(a_j)| \leqslant D(\frac{1}{n})$.

Similarly $|z(s)-z(a_{j+1})| \leqslant D(\frac{1}{n})$. Consequently,

$$(4.2.9) \quad |z(s)-\hat{z}(s)| \leqslant n\left[|(a_{j+1}-s)D(\tfrac{1}{n})| + |(s-a_j)D(\tfrac{1}{n})|\right]$$

$$= nD(\tfrac{1}{n}).|a_{j+1}-a_j| = D(\tfrac{1}{n}).$$

If $z = U(x)$, $x \in \hat{C} = X$ then

$$(4.2.10) \quad |z(s)| = |(Ux)(s)| = \left|\int_0^1 h(s,t)x(t)\,dt\right|$$

$$\leqslant \int_0^1 |h(s,t)x(t)|\,dt = \|z\|$$

and

$$|z(s)-z(s')| \leqslant \int_0^1 |h(s,t)-h(s',t)|\,|x(t)|\,dt$$

$$\leqslant D_s(\delta) \int_0^1 |x(t)|\,dt$$

$$= D_s(\delta)\,\|x\| \quad \text{where } |\sigma| = |s-s'| \leqslant \delta \ .$$

Now, if we take $D(\delta) = D_s(\delta)\,\|x\|$, then $D(\frac{1}{n}) = D_s(\frac{1}{n})\,\|x\|$

which together with (4.2.9) give $|z(s)-\hat{z}(s)| \leqslant D_s(\frac{1}{n})\,\|x\|$.

Hence, from (4.2.10) we get $\|U(x)-\hat{z}\| \leqslant D_s(\frac{1}{n})\|x\|$.

Definition 2.1.2 is satisfied with $\alpha_1 = D_s(\frac{1}{n})$.

By the same argument as in the case of (4.2.9), for $y \in X$ there is $\hat{y} \in \hat{X}$ such that

$$\|y-\hat{y}\| \leqslant \bar{D}(\tfrac{1}{n}) \ .$$

Thus the condition of definition 2.1.3 is satisfied with

$$\alpha_2 = \frac{1}{\|y\|}\,\bar{D}(\tfrac{1}{n}) \ .$$

Since $h(s,t)$, $y(s)$ are continuous by assumption, $D_t(\frac{1}{n})$, $D_s(\frac{1}{n})$, $\bar{D}(\frac{1}{n}) \longrightarrow 0$ as $n \longrightarrow \infty$. This implies that $\bar{\alpha}$, α_1, $\alpha_2 \longrightarrow 0$ as $n \longrightarrow \infty$ while $\|A\| = \|A_o\| = \|A_o^{-1}\| = 1$.

Consequently, the general theory of approximation is applicable, and from theorem 3.2.3 if λ is not characteristic value of equation (4.2.1) then the system (4.2.3) is soluble for sufficiently large n. Moreover, the sequence of the approximate solutions $A_o^{-1}\bar{x}_n^*$ converges to the exact solution.

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